A kinetic model for grain growth

DISSERTATION

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)

im Fach Mathematik

eingereicht an der

Mathematisch-Naturwissenschaftlichen Fakultät II
der Humboldt-Universität zu Berlin

von
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geboren am 21. Dezember 1977 in Bonn

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Tag der Verteidigung: 17. September 2007

Abstract

The subject matter of this thesis is a detailed analysis of the self–consistent kinetic model for grain growth introduced by Fradkov [5]. The model is based on the *von Neumann–Mullins law* describing the change of area of grains according to their topological class, i.e. the number of edges they have. Topological events are performed by coupling terms between equations for the number densities of different topological classes. The resulting system of transport equations is infinite–dimensional with a tridiagonal coupling structure. Self–consistency of this kinetic model is achieved by introducing a coupling's weight Γ making the equations nonlinear and nonlocal in space.

We start with an introduction in the first chapter. Afterwards in the second chapter we derive Fradkov's model and carry out formal calculations to illustrate self–consistency.
In the third chapter we present a–priori calculations mainly allowing us to bound the nonlinearity Γ. This enables us to prove existence and uniqueness of solutions to finite–dimensional systems in the first part of the fourth chapter.
Further bounds on the number densities established in the fifth chapter allow for passing to the limit concerning the number of equations in the second part of the fourth chapter. Therefore we prove existence of solutions to the infinite–dimensional system by a suitable approximation procedure. Uniqueness and continuous dependence on the data is then provided by energy methods.
The sixth chapter focusses on long–time behaviour and mainly on stationary solutions of a rescaled system as candidates for self–similar solutions. Finally we prove *Lewis' law* asymptotically.

Keywords:
grain growth, kinetic model, infinite–dimensional, hyperbolic

Zusammenfassung

In dieser Arbeit wird eine detaillierte Analysis des konsistenten kinetischen Modells zum Kornwachstum von Fradkov [5] durchgeführt. Dieses Modell beschreibt — basierend auf dem *von Neumann–Mullins Gesetz* — die Flächenänderung eines Korns abhängig von seiner Topologieklasse, d.h. der Anzahl der Kanten. Topologieänderungen werden durch Kopplungsterme zwischen den Gleichungen für die Anzahldichten der verschiedenen Topologieklassen beschrieben. Daraus resultiert ein unendlich–dimensionales System von Transportgleichungen mit tridiagonaler Kopplungsstruktur. Durch eine spezielle Wahl des Kopplungsgewichts Γ, welche die Gleichungen nichtlinear und räumlich nichtlokal macht, wird das Modell konsistent.

Nach einer Einführung wird das Modell von Fradkov im zweiten Kapitel hergeleitet; formale Rechnungen zeigen die Konsistenz des Modells auf.
Im dritten Kapitel wird das Kopplungsgewicht Γ a priori beschränkt. Dadurch kann im ersten Teil des vierten Kapitels Existenz und Eindeutigkeit von Lösungen für endlich–dimensionale Systeme gezeigt werden.
Weitere Schranken an die Anzahldichten im fünften Kapitel ermöglichen den Grenzübergang hinsichtlich der Anzahl der Gleichungen im zweiten Teil des vierten Kapitels. Die Existenz von Lösungen des unendlich–dimensionalen Systems wird somit über eine geeignete Approximation gezeigt. Energiemethoden liefern Eindeutigkeit und stetige Abhängigkeit von den Daten.
Im sechsten Kapitel wird das Langzeitverhalten untersucht. Besonderes Augenmerk liegt dabei auf stationären Lösungen eines reskalierten Systems als Kandidaten für selbstähnliche Lösungen. Abschließend wird das *Lewis'sche Gesetz* asymptotisch verifiziert.

Schlagwörter:
Kornwachstum, kinetisches Modell, unendlich–dimensional, hyperbolisch

Für Ingrid, die immer an mich geglaubt hat.

Contents

Chapter 1

Introduction

Most technologically useful materials are polycrystalline aggregates, composed of a huge number of crystallites, called grains, separated by so–called grain boundaries. The application of such materials covers many scales, from steel girders for power poles to base plates of microprocessors. Important material properties like fracture, toughness, or conductivity are determined by the polycrystalline microstructure, i.e. the sizes, shapes, orientation, and arrangement of grains. Unfortunately such materials undergo a temperature controlled aging process leading to a coarsening of the grain structure and therefore inducing changes in mechanical, electrical, optical, and magnetic properties of the material. For further details we refer to the review articles by Fradkov and Udler [6] and Thompson [20].

Different approaches for modelling grain growth in two space dimensions are established in the literature. In Monte–Carlo models [1, 2] the microstructure is mapped onto a discrete lattice. The kinetics of the boundary motion are simulated by employing a Monte–Carlo technique for moving these lattice points. An attractive feature of this model is the simple handling of topological events like grain boundary flipping and grain disappearance.
Using boundary tracking models based on partial differential equations [12, 14] offers an attractive alternative to Monte–Carlo models since they deal with quantities of lower dimension. A disadvantage arises as topological changes require extra treatment. The idea of reducing interfacial energy as driving force in grain growth is also carried on in vertex models where movement of grain boundaries is projected onto the triple–junctions [11, 9]. Another class are mean field models for grain growth [20, Section IV]. In the sequel we focus on kinetic models [5, 15, 4] based on the *von Neumann–Mullins law*. Such models consider time–dependent distribution functions for the grain areas and the number of sides per grain. Grain areas change ac-

cording to the *von Neumann–Mullins law*, topological changes are performed by collision operators. Fradkov was the first to develop a model of this type [5] and up to now there is no analytic treatment of such a model. Therefore this thesis establishes a rigorous theory for the arising infinite–dimensional system of transport equations with nonlocal weight making the equations nonlinear.

In Chapter 2 we derive Fradkov's self–consistent model and verify certain natural relations by formal calculations.

Chapter 3 provides us with some necessary a–priori calculations mainly enabling us to bound the solution and the coupling's weight. Furthermore we indicate how to transform the system via the method of characteristics and prove non–negativity of solutions. The key features of this chapter are a (in the continuous variable, the grain area) constant supersolution decaying exponentially with respect to the discrete variable (the topological class) and an argument preventing the total mass from dropping down to zero within finite times based on considerations regarding the maximum annihilation speed for disappearing grains.

We first prove existence of solutions to finite–dimensional systems by a fixed point argument in Chapter 4. Then we pass to the limit proving existence of solutions to the infinite–dimensional system using the bounds on the solution which we obtained Chapter 3 and Chapter 5 and achieving compactness by Arzela–Ascoli. We also prove uniqueness and continuous dependence on the data by energy methods.

In Chapter 5 we prove that no mass runs off at infinity neither with respect to the continuous variable nor to the discrete one. Our main idea is to exploit an interplay between the decay concerning the discrete variable and the one regarding the continuous variable by using a bounding frame that grows in time. Furthermore we verify the natural relations treated in Chapter 2 in the infinite–dimensional case, too.

Chapter 6 is dedicated to the long–time behaviour of solutions. Besides a characterization of stationary solutions we focus our attention on self–similar solutions. We rescale the equations, consider stationary solutions, and solve the resulting system of ordinary differential equations formally. Furthermore we deal with *Lewis' law* in self–similar variables asymptotically.

[The curtain rises.]

Chapter 2

Derivation of a consistent kinetic model

The purpose of this chapter is to derive a kinetic model for two–dimensional grain growth proposed by Fradkov [5, 8]. Furthermore we carry out formal calculations to illustrate self–consistency of this model.

2.1 Preliminaries

We start our considerations with some remarks on a well–established model for grain growth leading to the well–known *von Neumann–Mullins law* which is the foundation of our kinetic model.

2.1.1 Motion by mean curvature and equilibrium of forces at triple junctions

Mean curvature flow coupled with equilibrium of forces at triple junctions is a widely accepted model for two–dimensional grain growth (cf. [3, 12, 14]). In the sequel we briefly recall the basic model. Therefore our objects are networks of curves which meet in triple junctions. (We will refer to this as the *triple junction condition.*) Since we are mainly interested in settings with large numbers of grains and not so much in the influence of boundary conditions, we consider one–periodic spatial networks. Furthermore we restrict ourselves to the case of isotropic surface energies and assume the mobility of the triple junctions to be infinitely fast compared to the mobility of the grain boundaries. (We also assume the mobilities of the grain boundaries to be equal.) Now we can evolve the network due to mean curvature flow as long as no grain boundaries vanish. Furthermore we have to take the *Her-*

ring condition [10] into account that prescribes equilibrium of forces at triple junctions. In the isotropic case this just means that the curves meet in an angle of $2\pi/3$. In general *Young's relations* imply that the ratios of surface energies and the sine of the opposite angle are equal [17]. Observing that mean curvature flow has a natural interpretation as a gradient flow [19, 9], the *Herring condition* arises as a natural boundary condition (coming up via an integration by parts) when computing the differential of the associated L^2 energy (cf. Appendix B).

2.1.2 von Neumann–Mullins law

Under the assumptions stated above (isotropic surface energy, equal mobility of grain boundaries, and infinite mobility of triple junctions), one can use the concept of a network evolving by mean curvature flow coupled with equilibrium of forces at triple junctions to derive a law of motion for the area of a single grain with n edges [16], known as the *von Neumann–Mullins law*:

$$\frac{d}{dt} a\left(t\right) = M\sigma\frac{\pi}{3}\left(n-6\right) \tag{2.1}$$

M denotes the mobility of the grain boundaries and σ the surface tension. The proof can be done by a direct geometric computation using motion by curvature of the grain boundaries and the prescribed jumps of the outer normal by $2\pi/3$ at triple junctions (cf. Appendix C).

The *von Neumann–Mullins law* implies that grains with less than six edges shrink, those with more than six grow, and such with exactly six edges retain their area (possibly not their shape).

2.1.3 Topological changes

The evolution sketched in Subsection 2.1.1 is well–defined until two vertices on a grain boundary collide, after which topological rearrangements may take place. This happens when either an edge or a whole grain vanishes. In the first case an unstable fourfold vertex is produced, which immediately splits up again, usually in such a way that two new vertices are connected by a new edge. In this case, two neighbouring grains decrease their topological class, whereas the two other grains increase it (Figure 2.1). The second case causing topological rearrangements is grain vanishing. Each grain vanishing is accompanied by disappearance of two vertices and three edges. Due to the *von Neumann–Mullins law* we only take grains with topological class $2 \leq n \leq 5$ into account. Grains with $n = 2$ and $n = 3$ vanish in a single

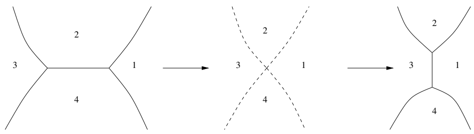

Figure 2.1: Neighbour switching

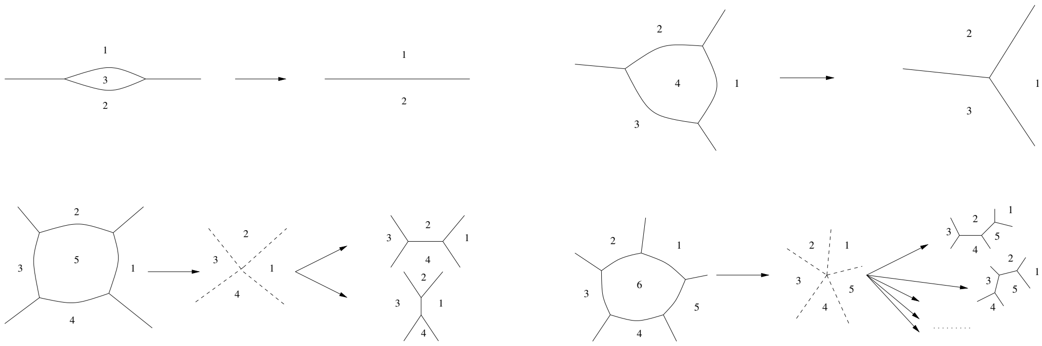

Figure 2.2: Grain vanishing

possible way. For $n = 4$ we observe two topologically distinguishable possibilities and for $n = 5$ even five possible local configurations (Figure 2.2). For further details on the resulting topological classes associated with adopted topological configurations after vanishing events we refer to the review article by Fradkov and Udler [6].

At this stage it is completely unclear by which mechanism a specific topological configuration is selected within switching or after vanishing events. A natural idea is to assume that this selection process is driven by the same tendency to reduce energy the whole network structure evolves by. One possibility is to compute all possible local configurations and select the one that minimizes energy locally in the best way [9].

2.2 One–particle distribution

In common with Fradkov [5, 6] we introduce a number density $f_n(a,t)$ that measures the number of grains with topological class n and area a at time t. Using the *von Neumann–Mullins law* (2.1) we can describe the evolution of f by transport equations

$$\begin{aligned} \partial_t f_n(a,t) + (n-6)\,\partial_a f_n(a,t) &= 0 \\ f_n(a,0) &= g_n(a) \end{aligned} \qquad n \geq 2 \tag{2.2}$$

as long as no topological rearrangements take place. Note that the factors M and σ in (2.1) are constants and therefore were scaled out in the same way as $\pi/3$.

To model topological changes we introduce a collision term $(\tilde{J}f)_n$ on the r.h.s. of (2.2) coupling the equations. We define *topological fluxes* X_n^+ and X_n^- denoting the flux from class n to $n+1$ and from n to $n-1$ respectively.

$$\left(\tilde{J}f\right)_n = X_{n-1}^+ + X_{n+1}^- - X_n^+ - X_n^-$$

In the sequel we state the *'gas' approximation* of the collision kernel by Fradkov [5]. Each grain of topological class n in a two–dimensional one–periodic network is bounded by n edges and features n triple junctions on it's boundary. We can identify three possible events causing transitions of the grain from one topological class to another:

- switching of an edge (which is part of the grain boundary) leading to a transition from topological class n to $(n-1)$
- switching of an outgoing edge causing a transition from topological class n to $(n+1)$
- vanishing of a neighbour grain, which causes a transition from topological class n to $(n-1)$

Here we ignore that the topological class of a grain is lowered by two if the neighbouring annihilated grain was a lense, i.e. had topological class $n = 2$. Fradkov and Udler argue [6] that such an event takes place only very rarely as the number of lenses itself is already very small.

Now we make a strong assumption by neglecting correlation effects when switching or vanishing events take place, i.e. we assume that the probability for the occurrence of topological changes is only proportional to n (and independent of a and neighbour correlations). Furthermore we assume that the probabilities of switching are equal for all boundaries in the system. This implies that for any given grain the two switching events described above are equally probable. Due to these assumptions the topological fluxes can be expressed as follows:

$$X_n^+ = \Gamma\beta\, n f_n\,, \quad X_n^- = \Gamma\,(\beta+1)\, n f_n$$

Note that $\beta \in (0,2)$ is a free parameter in this model describing the ratio between "symmetric" and "asymmetric" topological events. The bounds on

β are needed (cf. Lemma 2.3) to estimate the nonlinearity $\Gamma = \Gamma(f)$ which we determine later on (2.5) to achieve self–consistency (cf. Lemma 2.2). Now the collision terms (we will call them coupling terms in the following) read as

$$\Gamma(f)\left[(\beta+1)(n+1) f_{n+1} - (2\beta+1) n f_n + \beta (n-1) f_{n-1}\right], \quad n > 2$$

and for $n = 2$ we have

$$\Gamma(f)\left[(\beta+1) 3 f_3 - 2\beta f_2\right]$$

to ensure the *zero balance property* (see (2.9) below). At this stage we are able to state the complete kinetic model. Using the notation $\sum_n$ means summing up over all admissible n, i.e. $\sum_{n\geq 2}$ in the setting of Subsection 2.2.1 and $\sum_{n=2}^{n_0}$ in the setting of Subsection 2.2.2.

2.2.1 Infinite system

The equations stated in the sequel are mainly the same as in the work of Fradkov [5, 8, 6]. The coupling term $(Jf)_2$ differs and we do not neglect $\int f_2\, da$ within the denominator of $\Gamma(f(t))$.

$$\begin{aligned} \partial_t f_n(a,t) + (n-6)\,\partial_a f_n(a,t) &= \Gamma(f(t))\,(Jf)_n(a,t) \\ f_n(a,0) &= g_n(a) \end{aligned} \qquad n \geq 2 \tag{2.3}$$

$$\begin{aligned} (Jf)_n(a,t) &= \beta\left((n+1) f_{n+1}(a,t) - 2n f_n(a,t) + (n-1) f_{n-1}(a,t)\right) \\ &\quad + (n+1) f_{n+1}(a,t) - n f_n(a,t) \\ (Jf)_2(a,t) &= \beta\left(3 f_3(a,t) - 2 f_2(a,t)\right) + 3 f_3(a,t) \\ &\text{where} \quad \beta \in (0,2), \quad n > 2 \end{aligned} \tag{2.4}$$

The coupling's weight Γ making the equations nonlinear (and nonlocal in space) is chosen as

$$\Gamma(f(\cdot,t)) = \frac{\sum\limits_{n\geq 2} (n-6)^2 f_n(0,t)}{\sum\limits_{n\geq 2} n \int\limits_0^\infty f_n(a,t)\,da - 2(\beta+1)\int\limits_0^\infty f_2(a,t)\,da} \tag{2.5}$$

which ensures the preservation of the triple junction condition as stated in Lemma 5.4. As boundary conditions we set

$$f_n(0,t) = 0 \quad n > 6 \tag{2.6}$$

for $0 < t < \infty$ ensuring that no additional mass is transported from the negative half–axis to the positive one. This means no additional grains can be created.

2.2.2 Finite system

Within our proof we will also use the finite–dimensional analogue of (2.3).

$$\begin{aligned} \partial_t f_n(a,t) + (n-6)\,\partial_a f_n(a,t) &= \Gamma(f(t))\,(Jf)_n(a,t) \\ f_n(a,0) &= g_n(a) \end{aligned} \qquad 2 \le n \le n_0 \tag{2.7}$$

$$\begin{aligned} (Jf)_{n_0}(a,t) &= \beta\left(-n_0 f_{n_0}(a,t) + (n_0-1)\,f_{n_0-1}(a,t)\right) - n_0 f_{n_0}(a,t) \\ (Jf)_n(a,t) &= \beta\left((n+1)\,f_{n+1}(a,t) - 2n f_n(a,t) + (n-1)\,f_{n-1}(a,t)\right) \\ &\quad + (n+1)\,f_{n+1}(a,t) - n f_n(a,t) \\ (Jf)_2(a,t) &= \beta\left(3f_3(a,t) - 2f_2(a,t)\right) + 3f_3(a,t) \\ &\text{where} \quad \beta \in (0,2)\,, \quad 2 < n < n_0 \end{aligned} \tag{2.8}$$

The coupling defined in (2.8) operates pointwise and has the important *zero balance property* (2.9) reflecting that no grains can be generated or annihilated by topological rearrangements.

$$\sum_n (Jf)_n(a,t) = 0 \tag{2.9}$$

The following choice of $\Gamma : \left[L^1 \cap C^0\right]^{n_0-1} \to \mathbb{R}$ in (2.10) reflects the preservation of the triple junction condition (cf. Lemma 2.2) and will be derived in Subsection 2.3.2.

$$\Gamma(f(t)) = \frac{\sum_n (n-6)^2 f_n(0,t)}{\sum_n n \int_0^\infty f_n(a,t)\,da - 2(\beta+1)\int_0^\infty f_2(a,t)\,da + n_0\beta \int_0^\infty f_{n_0}(a,t)\,da} \tag{2.10}$$

Furthermore we state

$$f_n(0,t) = 0 \quad n > 6 \tag{2.11}$$

for $0 < t < \infty$ as boundary conditions.

2.3 Bounded and conserved quantities

Within this section we will carry out some formal calculations – assuming a pointwise non–negative solution f of (2.7) exists – to identify certain bounded and conserved quantities. Calculations concerning pointwise non–negative solutions of (2.3) are formally carried out in the same way and are therefore not stated here.

From now on we consider solutions f to the finite system (2.7) which are bounded and continuously differentiable w.r.t. a and t separately. Furthermore the nonlinearity $\Gamma(f(t))$ shall be bounded, too.

2.3.1 Total number of grains

If equations (2.7) reflect a coarsening process, it should be clear that the total number of grains decreases in time. To be more precise we define $N(t)$ by simply counting all grains at time t.

Definition 2.1 (total number of grains) *We call*

$$N(t) = \sum_n \int_0^\infty f_n(a,t)\,da$$

the total number of grains at time t.

Lemma 2.1 *If a solution f to non–negative initial data g exists, then we have*

$$\frac{d}{dt}N(t) = \sum_n (n-6)\, f_n(0,t)$$

for all finite times.

Proof of Lemma 2.1
Differentiating $N(t)$ and using (2.7) gives us

$$\begin{aligned}
\frac{d}{dt}N(t) &= \frac{d}{dt}\sum_n \int_0^\infty f_n(a,t)\,da \\
&= \sum_n \int_0^\infty \partial_t f_n(a,t)\,da \\
&= -\sum_n (n-6)\int_0^\infty \partial_a f_n(a,t)\,da + \Gamma(f(t))\int_0^\infty \underbrace{\sum_n (Jf)_n(a,t)}_{=0}\,da \\
&= \sum_n (n-6)\, f_n(0,t)
\end{aligned}$$

as f takes zero values at ∞ w.r.t. a due to the exponentially decaying supersolution (cf. Lemma 3.3) and by using the *zero balance property* (2.9) of $\sum_n (Jf)_n(a,t)$. *q.e.d.*

Corollary 2.1 (decrease of total number of grains) *Under the natural assumptions of Lemma 2.1 we have*

$$\frac{d}{dt}N(t) \le 0$$

for all finite times.

Proof of Corollary 2.1
Due to the boundary conditions (2.11) to (2.7) Lemma 2.1 reads

$$\frac{d}{dt} N(t) = -\sum_{n=2}^{5} (6-n) \underbrace{f_n(0,t)}_{\geq 0} \leq 0$$

as f is pointwise non–negative (cf. Lemma 3.1). *q.e.d.*

2.3.2 Triple junction condition

Another important feature that solutions to (2.7) should reflect is the validity of the triple junction condition. Hence we will check whether Euler's polyhedral formula holds during the evolution.

Proposition 2.1 (polyhedral formula) *For a periodic polygon Euler's polyhedral formula reads*

$$V + F - E = 0 \tag{2.12}$$

where V denotes the number of vertices, F the number of facets, and E the number of edges.

Proof of Proposition 2.1
Poincaré's version of the polyhedral formula reads

$$V + F - E = \chi(g)$$

where g is the genus of the surface and

$$\chi(g) = 2 - 2g$$

the Euler characteristic. Considering a periodic polyhedron means looking at a polyhedron on a torus. A torus has genus $g = 1$ and therefore the Euler characteristic is $\chi(g) = 0$. *q.e.d.*

Now we translate the polyhedral formula into our setting.

Proposition 2.2 *The polyhedral formula reads*

$$\sum_n (n-6) \int_0^\infty f_n(a,t)\, da = 0 \tag{2.13}$$

for solutions to (2.7) *(under triple junction condition).*

Proof of Proposition 2.2
The number of facets F of our periodic network is given by

$$F(t) = \sum_n \int_0^\infty f_n(a,t)\,da$$

and the number of edges E can be computed via

$$E(t) = \frac{1}{2}\sum_n n \int_0^\infty f_n(a,t)\,da$$

at any time t. The number of vertices V is given by

$$V(t) = \frac{2}{3}E(t) = \frac{1}{3}\sum_n n \int_0^\infty f_n(a,t)\,da$$

as we constrain that edges shall only meet in triple junctions. Plugging $F(t)$, $E(t)$, and $V(t)$ into (2.12) completes the proof. *q.e.d.*

We treat the validity of the polyhedral formula as evidence that the triple junction condition holds. This gives rise to the following lemma.

Lemma 2.2 *If the initial data g satisfy*

$$\sum_n (n-6) \int_0^\infty g_n(a)\,da = 0$$

then the polyhedral formula (2.13) *for solutions f of* (2.7)*, that is*

$$\sum_n (n-6) \int_0^\infty f_n(a,t)\,da = 0\,,$$

is satisfied for all times $0 < t < \infty$.

Proof of Lemma 2.2
We carry out the proof by differentiating the polyhedral formula (2.13)

$$\begin{aligned}
&\frac{d}{dt}\sum_n (n-6)\int_0^\infty f_n(a,t)\,da \\
=&\sum_n (n-6)\int_0^\infty \partial_t f_n(a,t)\,da \\
=&-\sum_n (n-6)^2 \int_0^\infty \partial_a f_n(a,t)\,da + \sum_n (n-6)\int_0^\infty \Gamma(f(t))\,(Jf)_n(a,t)\,da \\
=&\sum_n (n-6)^2 f_n(0,t) + \Gamma(f(t))\sum_n n \int_0^\infty (Jf)_n(a,t)\,da
\end{aligned}$$

and using the *zero balance property* (2.9) $\sum_n (Jf)_n (a,t) = 0$ to obtain the last equality in the above calculation.

Now we will focus our attention on the weighted sum of the coupling terms omitting the arguments a and t of f.

$$\begin{aligned}\sum_{n=2}^{n_0} n\,(Jf)_n &= \beta \sum_{n=3}^{n_0} n\,(n-1)\, f_{n-1} + (\beta+1) \sum_{n=2}^{n_0-1} n\,(n+1)\, f_{n+1} \\ &\quad - \beta \sum_{n=2}^{n_0-1} n^2 f_n - (\beta+1) \sum_{n=3}^{n_0} n^2 f_n \\ &= -\left(\sum_{n=2}^{n_0} n f_n - 2\,(\beta+1)\, f_2 + n_0 \beta f_{n_0} \right)\end{aligned}$$

The above computations are done by using $n\,(n-1) = (n-1)^2 + (n-1)$ and $n\,(n+1) = (n+1)^2 - (n+1)$ and some index shifts in the resulting sums. By the choice of $\Gamma(f(t))$ in equation (2.10)

$$\Gamma(f(t)) = \frac{\sum\limits_n (n-6)^2 f_n(0,t)}{\sum\limits_n n \int\limits_0^\infty f_n(a,t)\,da - 2\,(\beta+1) \int\limits_0^\infty f_2(a,t)\,da + n_0 \beta \int\limits_0^\infty f_{n_0}(a,t)\,da}$$

we finally get

$$\begin{aligned}&\frac{d}{dt} \sum_n (n-6) \int_0^\infty f_n(a,t)\,da \\ &\qquad = \sum_n (n-6)^2 f_n(0,t) - \Gamma(f(t)) \left(\sum_n n \int_0^\infty f_n(a,t)\,da \right) \\ &\qquad\quad + \Gamma(f(t)) \left(2\,(\beta+1) \int_0^\infty f_2(a,t)\,da - n_0 \beta \int_0^\infty f_{n_0}(a,t)\,da \right) \\ &\qquad = 0\end{aligned}$$

q.e.d.

With this knowledge we are able to establish a first estimate on $\Gamma(f(t))$.

Lemma 2.3 *If f is a solution to* (2.7) *and its initial data satisfy the polyhedral formula* (2.13), *then*

$$\Gamma(f(t)) \leq -c\,\frac{\dot{N}(t)}{N(t)}$$

where $c > 0$ is a uniform constant.

Proof of Lemma 2.3
The numerator of $\Gamma(f(t))$ can be bounded from above via

$$\sum_n (n-6)^2 f_n(0,t) \leq -4 \sum_{n=2}^{5} (n-6) f_n(0,t) = -4\dot{N}(t)$$

as $f_n(0,t) = 0$ for $n \geq 6$ due to the boundary conditions (2.11).
We will proceed by bounding the denominator of $\Gamma(f(t))$ from below by using Lemma 2.2.

$$\begin{aligned}
&\sum_n n \int_0^\infty f_n(a,t)\, da - 2(\beta+1) \int_0^\infty f_2(a,t)\, da + n_0\beta \int_0^\infty f_{n_0}(a,t)\, da \\
&\geq \sum_n 6 \int_0^\infty f_n(a,t)\, da - 2(\beta+1) \int_0^\infty f_2(a,t)\, da \\
&\geq \underbrace{(6-2(\beta+1))}_{>0} \sum_n \int_0^\infty f_n(a,t)\, da = c' N(t)
\end{aligned}$$

Note $c' > 0$ as $\beta \in (0,2)$. *q.e.d.*

2.3.3 Total covered area

With the result of Lemma 2.2 we are able to state the conservation of total covered area $A(t)$.

Definition 2.2 (total covered area) *We call*

$$A(t) = \sum_n \int_0^\infty a f_n(a,t)\, da$$

the total covered area at time t.

Lemma 2.4 *If the initial data g satisfy*

$$\sum_n (n-6) \int_0^\infty g_n(a)\, da = 0$$

then

$$\frac{d}{dt} A(t) = 0$$

i.e. total covered area is conserved.

Proof of Lemma 2.4
Differentiating $A(t)$ and using (2.7) gives us

$$\begin{aligned}\frac{d}{dt}A(t) &= \frac{d}{dt}\sum_n \int_0^\infty a f_n(a,t)\,da = \sum_n \int_0^\infty a\partial_t f_n(a,t)\,da \\ &= -\sum_n (n-6)\int_0^\infty a\partial_a f_n(a,t)\,da + \Gamma(f(t))\int_0^\infty a \underbrace{\sum_n (Jf)_n(a,t)\,da}_{=0} \\ &= \sum_n (n-6)\int_0^\infty f_n(a,t)\,da\end{aligned}$$

via an integration by parts and by using the zero balance property (2.9) of the sum of the coupling terms $\sum_n (Jf)_n(a,t)$; f takes zero values at ∞ w.r.t. a (cf. Lemma 3.3).
As the initial data satisfy the polyhedral formula (2.13), Lemma 2.2 gives us the result. *q.e.d.*

During a coarsening process the average area of grains increases in time. This can also be concluded easily from the above inspection of $N(t)$ and $A(t)$.

Definition 2.3 (mean grain area) *We call*

$$M(t) = \frac{A(t)}{N(t)}$$

the mean grain area at time t.

Corollary 2.2 *If the initial data satisfy the polyhedral formula* (2.13)*, then*

$$\frac{d}{dt}M(t) \geq 0$$

i.e. the mean grain area of solutions f of (2.7) *is increasing in time.*

Proof of Corollary 2.2
Differentiating $M(t)$ yields

$$\frac{d}{dt}M(t) = -c\frac{\dot{N}(t)}{N^2(t)} \geq 0$$

as $A(t)$ is constant (Lemma 2.4) for all finite times and $N(t)$ decreasing (Lemma 2.1) in time. *q.e.d.*

Chapter 3

A–priori calculations

All calculations within this chapter can be applied to solutions of either the infinite system (2.3) stated in Subsection 2.2.1 or the finite system (2.7) stated in Subsection 2.2.2 except for Lemma 3.3 which obviously makes sense for the finite system only.

From now on we consider solutions f to the finite system (2.7) which are bounded and continuously differentiable w.r.t. a and t separately. Furthermore the nonlinearity $\Gamma(f(t))$ shall be bounded, too.

3.1 Characteristics

A first step to construct solutions to (2.7) is to transform the given system of transport equations into a system of integral equations. This will enable us to use a fixed point argument later on. The transformation will be done via the method of characteristics.

Proposition 3.1 (integral equations) *The time–integrated version of the system* (2.7)

$$\begin{aligned} \partial_t f_n(a,t) + (n-6)\,\partial_a f_n(a,t) &= \Gamma(f(t))\,(Jf)_n(a,t) \\ f_n(a,0) &= g_n(a) \end{aligned} \qquad 2 \le n \le n_0$$

is given by

$$\begin{aligned} f_n(a,t) &= g_n(a-(n-6)\,t) \\ &\quad + \int_0^t \Gamma(f(s))\,(Jf)_n(a-(n-6)(t-s),s)\,ds \qquad (3.1) \\ f_n(a,0) &= g_n(a) \qquad\qquad 2 \le n \le n_0 \end{aligned}$$

for $a \in (0,\infty)$ and $t \in (0,\infty)$. We set $g_n(a) = 0$ for $a < 0$. The coupling $(Jf)_n(a,t)$ is defined by equations (2.8) *and the nonlinearity $\Gamma(f(t))$ is given by equation* (2.10) *in Subsection 2.2.2.*
Note that we set $f_n(\alpha,t) = 0$ if the argument α is negative in the formulas above.

Proof of Proposition 3.1
For an arbitrary $n \in \{2,...,n_0\}$ we set

$$z(s) = f_n(a + s(n-6), t+s)$$

for any $a \geq s(n-6)$ and $t + s \geq 0$. Differentiating $z(s)$ yields

$$\begin{aligned}\dot{z}(s) &= \partial_t f_n(a + s(n-6), t+s) + (n-6)\,\partial_a f_n(a + s(n-6), t+s)\\ &= \Gamma(f(t+s))\,(Jf)_n(a + s(n-6), t+s)\end{aligned}$$

by using (2.7) for the last equality. Now we observe

$$\begin{aligned}f_n(a,t) - g_n(a-(n-6)t) &= z(0) - z(-t)\\ &= \int_{-t}^{0} \dot{z}(s)\,ds\\ &= \int_{-t}^{0} \Gamma(f(t+s))\,(Jf)_n(a + s(n-6), t+s)\,ds\\ &= \int_{0}^{t} \Gamma(f(s))\,(Jf)_n(a-(n-6)(t-s), s)\,ds\end{aligned}$$

which completes the proof. *q.e.d.*

Remark 3.1 (pointwise coupling) *As the coupling acts in a pointwise way we shall clarify how to evaluate $(Jf)_n(a-(n-6)(t-s), s)$, namely by ignoring the dependence of $[a-(n-6)(t-s)]$ on n w.r.t. coupling.*

$$\begin{aligned}&(Jf)_n(a-(n-6)(t-s), s)\\ &\quad= \beta\Big((n+1)\,f_{n+1}(a-(n-6)(t-s), s)\\ &\qquad - 2nf_n(a-(n-6)(t-s), s) + (n-1)\,f_{n-1}(a-(n-6)(t-s), s)\Big)\\ &\qquad + (n+1)\,f_{n+1}(a-(n-6)(t-s), s) - nf_n(a-(n-6)(t-s), s)\end{aligned}$$

The terms $(Jf)_2(a-(n-6)(t-s), s)$ and $(Jf)_{n_0}(a-(n-6)(t-s), s)$ are treated in the same way.

Remark 3.2 *If f^ε is a sufficiently smooth solution to*

$$\begin{aligned} \partial_t f_n^\varepsilon(a,t) + (n-6)\,\partial_a f_n^\varepsilon(a,t) &= \Gamma(f(t))\,(Jf^\varepsilon)_n(a,t) + \varepsilon \\ f_n^\varepsilon(a,0) &= g_n(a) \end{aligned} \qquad 2 \le n \le n_0 \tag{3.2}$$

for $a \in (0,\infty)$ and $t \in (0,\infty)$, this system can be transformed into

$$\begin{aligned} f_n^\varepsilon(a,t) &= g_n(a-(n-6)\,t) \\ &\quad + \int_0^t \Gamma(f(s))\,(Jf^\varepsilon)_n\,(a-(n-6)\,(t-s)\,,s)\,ds + t\varepsilon \\ f_n^\varepsilon(a,0) &= g_n(a) \qquad\qquad 2 \le n \le n_0 \end{aligned} \tag{3.3}$$

for any $\varepsilon > 0$. Nonlinearity, coupling, and boundary conditions are defined in the same way as for f in equations (2.10)*,* (2.8)*, and* (2.11)*.*

3.2 Pointwise non–negativity

It is reasonable that a solution f to (2.7) with non–negative initial data should not become negative for any time t. This is stated in the following lemma.

Lemma 3.1 (non–negativity) *If f is a solution to* (2.7) *with non–negative initial data g, then f is non–negative ($f \ge 0$), i. e.*

$$f_n(a,t) \ge 0 \quad \forall\, n \in \{2,...,n_0\}$$

for any $a \in [\,0,\infty)$ and $t > 0$.

Proof of Lemma 3.1
First we observe

$$f_n(0,t) = 0 \quad n > 6$$

due to the boundary conditions (2.11). The cases

$$f_n(0,t) \ge 0 \quad 2 \le n \le 6$$

can be treated by considering f on $\mathbb{R}$ instead of $\mathbb{R}_+$ and setting

$$f_n(a,t) \equiv 0 \quad \text{for } a < 0 \text{ and } n > 6$$

for all t.
For keeping notation simple we only discuss $0 < a < \infty$ from now on.

We regard solutions f^ε to system (3.2) as mentioned in Remark 3.2 in the previous Section 3.1 which is similar to our original system (2.7) except that a positive ε is added to the right hand sides. If Γ is bounded, solutions f^ε to (3.2) can be constructed as fixed points of (3.3).
We consider the set of triples (n, a, t) where $f_n^\varepsilon(a,t)$ is negative and assume this set to be non–empty. We determine the time $\tau = \tau(\varepsilon)$ after which at least one $f_n^\varepsilon(a,t)$ becomes negative.

$$\tau(\varepsilon) := \inf\{t \mid \exists\, n, a \,:\, f_n^\varepsilon(a,t) < 0\}$$

We label the indices n and a that belong to $\tau = \tau(\varepsilon)$ by k and α, i.e. $\exists\, k, \alpha$ such that we have

$$f_k^\varepsilon(\alpha, \tau) = 0$$

by definition and

$$\partial_a f_k^\varepsilon(\alpha, \tau) = 0$$

because $f_n^\varepsilon(a,t)$ has a local minimum in a–direction at (k, α, τ) by construction. Note that neither k nor α can tend to infinity due to Lemmas 3.2 and 3.3.
Using (3.2) we find for the time–derivative of $f_k^\varepsilon(\alpha,\tau)$

$$\begin{aligned}
\partial_t f_k^\varepsilon(\alpha,\tau) &= -(n-6)\underbrace{\partial_a f_k^\varepsilon(\alpha,\tau)}_{=0} + \underbrace{\Gamma(f^\varepsilon(\tau))}_{\geq 0}(Jf^\varepsilon)_k(\alpha,\tau) + \varepsilon \\
&= \Gamma\Bigg((\beta+1)(k+1)\underbrace{f_{k+1}^\varepsilon(\alpha,\tau)}_{\geq 0} - (2\beta+1)\,k\underbrace{f_k^\varepsilon(\alpha,\tau)}_{=0} \\
&\qquad + \beta(k-1)\underbrace{f_{k-1}^\varepsilon(\alpha,\tau)}_{\geq 0}\Bigg) + \varepsilon \\
&\geq \varepsilon > 0
\end{aligned}$$

which is contradictory to the construction of (k, α, τ). The above calculation is valid for $2 < k < n_0$. Computations are similar in the case $k = 2$

$$\begin{aligned}
\partial_t f_2^\varepsilon(\alpha,\tau) &= +4\underbrace{\partial_a f_2^\varepsilon(\alpha,\tau)}_{=0} + \underbrace{\Gamma(f^\varepsilon(\tau))}_{\geq 0}(Jf^\varepsilon)_2(\alpha,\tau) + \varepsilon \\
&= \Gamma\Bigg((\beta+1)\,3\underbrace{f_3^\varepsilon(\alpha,\tau)}_{\geq 0} - \beta 2\underbrace{f_2^\varepsilon(\alpha,\tau)}_{=0}\Bigg) + \varepsilon \\
&\geq \varepsilon > 0
\end{aligned}$$

and also in the case $k = n_0$

$$\begin{aligned}\partial_t f^\varepsilon_{n_0}(\alpha,\tau) &= -(n_0-6)\underbrace{\partial_a f^\varepsilon_{n_0}(\alpha,\tau)}_{=0} + \underbrace{\Gamma(f^\varepsilon(\tau))}_{\geq 0}(Jf^\varepsilon)_{n_0}(\alpha,\tau)+\varepsilon \\ &= \Gamma\left(-(\beta+1)\,n_0\underbrace{f^\varepsilon_{n_0}(\alpha,\tau)}_{=0}+\beta(n_0-1)\underbrace{f^\varepsilon_{n_0-1}(\alpha,\tau)}_{\geq 0}\right)+\varepsilon \\ &\geq \varepsilon > 0\end{aligned}$$

leading to the same desired contradiction. Therefore the set

$$\{(n,a,t)\,|\,f^\varepsilon_n(a,t)<0\}$$

must be empty.
It remains to show that f^ε converges to f uniformly for $\varepsilon \to 0$. We define

$$z_\infty(t) := \sup_{n,a}|f^\varepsilon_n(a,t)-f_n(a,t)|$$

to measure the maximal distance between $f^\varepsilon(t)$ and $f(t)$. According to (3.3) we have

$$\begin{aligned}z_\infty(t) &\leq \sup_{n,a}\left|\int_0^t \Gamma(f(s))\,(Jf^\varepsilon)_n(a-(n-6)(t-s),s)\right. \\ &\qquad \left.-\Gamma(f(s))\,(Jf)_n(a-(n-6)(t-s),s)\,ds\right|+t\varepsilon \\ &\leq 4(\beta+1)\,n_0\sup_t|\Gamma(f(t))|\int_0^t\sup_{n,a}\left|f^\varepsilon_n(a-(n-6)(t-s),s)\right. \\ &\qquad \left.-f_n(a-(n-6)(t-s),s)\right|ds+t\varepsilon \\ &= \underbrace{4(\beta+1)\,n_0\sup_t|\Gamma(f(t))|}_{=:L}\int_0^t\underbrace{\sup_{n,a}|f^\varepsilon_n(a,t)-f_n(a,t)|}_{=z_\infty(s)}ds+t\varepsilon\end{aligned}$$

and so we have

$$z_\infty(t) \leq L\int_0^t z_\infty(s)\,ds+t\varepsilon \quad\Rightarrow\quad z_\infty(t)\leq \varepsilon\, t\exp(Lt)$$

by Gronwall's lemma. As $t \in (0,T]$, $T<\infty$, is arbitrary we get

$$\sup_t z_\infty(t) \leq \varepsilon\, T\exp(LT) \;\longrightarrow\; 0$$

the desired uniform convergence for $\varepsilon \to 0$. *q.e.d.*

3.3 Supersolutions

Within this section we will present supersolutions to the finite system of transport equations (2.7) introduced in Subsection 2.2.2. The first one will enable us to bound the numerator of $\Gamma(f(t))$ from above and is also applicable to the infinite system (2.3).

Definition 3.1 (supersolution) *We call a function $\bar{f}$ a supersolution to* (2.7) *if the initial data satisfy*

$$f_n(a,0) \le \bar{f}_n(a,0) \qquad \forall \quad n \in \{2,...,n_0\}, \quad 0 < a < \infty,$$

and if the inequality

$$\partial_t \bar{f}_n(a,t) + (n-6)\,\partial_a \bar{f}_n(a,t) - \Gamma(f(t))\left(J\bar{f}\right)_n(a,t) \ge 0 \tag{3.4}$$

holds for all $n \in \{2,...,n_0\}$, $0 < a < \infty$, and $t > 0$.

Note the dependence of $\Gamma = \Gamma(f(t))$. There is no $\Gamma\left(\bar{f}(t)\right)$ in the definition above; we consider $\Gamma(t)$ as a function of time once $f(t)$ is known.
If a function f solves (2.7) and $\bar{f}$ is a corresponding supersolution in the sense of Definition 3.1 we are able to bound f by $\bar{f}$ by considering a strict supersolution $\bar{f}\cdot\exp(\varepsilon t)$, using a comparison principle, and passing to the limit $\varepsilon \to 0$. To be more precise we state the following proposition.

Proposition 3.2 (comparison principle) *If we can bound the initial data of f strictly by the initial data of $\bar{f}$, namely*

$$f_n(a,0) < \bar{f}_n(a,0) \qquad \forall \quad n \in \{2,...,n_0\}, \quad 0 < a < \infty, \tag{3.5}$$

and if the strict inequality

$$\begin{aligned}\partial_t\left(\bar{f}_n(a,t) - f_n(a,t)\right) + (n-6)\,\partial_a\left(\bar{f}_n(a,t) - f_n(a,t)\right)&\\ - \Gamma(f(t))\left(J\left(\bar{f}_n(a,t) - f_n(a,t)\right)\right) &> 0\end{aligned} \tag{3.6}$$

holds for all $n \in \{2,...,N\}$, $0 < a < \infty$, and $t > 0$, then we can conclude that $\bar{f}$ bounds f pointwise for all positive times, namely

$$f_n(a,t) < \bar{f}_n(a,t) \qquad \forall \quad n \in \{2,...,n_0\}, \quad 0 < a < \infty, \quad t > 0. \tag{3.7}$$

Proof of Proposition 3.2
The proof is the same as the one of Lemma 3.1. *q.e.d.*

The following lemma presents a special supersolution which is constant in time and space. Therefore terms containing partial derivatives ∂_t and ∂_a of $\bar{f}$ will vanish and we can focus our attention on the effect of the coupling terms.

Lemma 3.2 (constant supersolution)

$$\bar{f}_n(a,t) = \frac{c}{\beta n}\left(\frac{\beta}{\beta+1}\right)^n \quad 2 \leq n \leq n_0 \tag{3.8}$$

is a supersolution to (2.7) *in the sense of Definition 3.1; c is a constant depending only on* $\sup_{n,a} f_n(a,0)$ *in such a way that* (3.5) *is satisfied.*

Proof of Lemma 3.2
In order to emphasize that $\bar{f}$ is exponentially decreasing in n we define γ as

$$\gamma = \log\left(1 + \frac{1}{\beta}\right) \tag{3.9}$$

and note $\gamma > 0$. Now $\bar{f}$ reads

$$\bar{f}_n(a,t) = \frac{c}{\beta n}\exp(-\gamma n) \quad 2 \leq n \leq n_0.$$

As $\bar{f}$ is constant in a and t we only have to check whether

$$-\Gamma(f(t))\left(J\bar{f}\right)_n(a,t) \geq 0$$

holds for all n. We only consider $\Gamma(f(t)) > 0$ as the case $\Gamma(f(t)) = 0$ is trivial and $\Gamma(f(t)) < 0$ cannot occur due to Lemmas 2.2 and 3.1 as $\beta \in (0,2)$. With a slight abuse of notation we set $c = 1$ instead of dividing by c and start examining the coupling terms.
$n = 2$:

$$\begin{aligned} -\left(J\bar{f}\right)_2(a,t) &= -3(\beta+1)\bar{f}_3(a,t) + 2\beta\bar{f}_2(a,t) \\ &= -\frac{\beta+1}{\beta}\exp(-3\gamma) + \exp(-2\gamma) \\ &\geq 0 \end{aligned}$$

The choice of γ in (3.9) is sufficient to ensure the desired non–negativity as we only need $1 + \frac{1}{\beta} \leq \exp(\gamma)$.
$n = n_0$:

$$\begin{aligned} -\left(J\bar{f}\right)_{n_0}(a,t) &= (\beta+1)n_0\bar{f}_{n_0}(a,t) - \beta(n_0-1)\bar{f}_{n_0-1}(a,t) \\ &= \frac{\beta+1}{\beta}\exp(-\gamma n_0) - \exp(-\gamma(n_0-1)) \\ &\geq 0 \end{aligned}$$

Again due to (3.9) we have $1 + \frac{1}{\beta} \geq \exp(\gamma)$ ensuring the non–negativity.
$2 < n < n_0$:

$$\begin{aligned}
-\left(J\bar{f}\right)_n(a,t) &= -\beta(n+1)\bar{f}_{n+1}(a,t) + 2\beta n\bar{f}_n(a,t) - \beta(n-1)\bar{f}_{n-1}(a,t) \\
&\quad - (n+1)\bar{f}_{n+1}(a,t) + n\bar{f}_n(a,t) \\
&= -\exp(-\gamma(n+1)) + 2\exp(-\gamma n) - \exp(-\gamma(n-1)) \\
&\quad - \frac{1}{\beta}\exp(-\gamma(n+1)) + \frac{1}{\beta}\exp(-\gamma n) \\
&= \exp(-\gamma n)\left(2 - \exp(\gamma) - \exp(-\gamma) + \frac{1}{\beta}(1 - \exp(-\gamma))\right) \\
&= \exp(-\gamma n)\left(-\exp(\gamma)(1 - \exp(-\gamma))^2 + \frac{1}{\beta}(1 - \exp(-\gamma))\right) \\
&\geq 0
\end{aligned}$$

The necessary non–negativity is again provided by (3.9).

$$\frac{1}{\beta} \geq \exp(\gamma)(1 - \exp(-\gamma)) \quad \Leftrightarrow \quad 1 + \frac{1}{\beta} \geq \exp(\gamma)$$

q.e.d.

In the following lemma we present an exponentially decaying (w.r.t. a) strict supersolution that allows the use of a comparison principle to show that solutions f of (2.7) must at least decay exponentially. This result is only true for finite n_0.

Lemma 3.3 (decaying supersolution)

$$\begin{aligned}
\bar{f}_n(a,t) &= \frac{c}{\beta n}\exp\left(t - \frac{a}{n_0} - \gamma n\right) \quad 2 \leq n \leq n_0 \\
\gamma &= \log\left(1 + \frac{1}{\beta}\right)
\end{aligned} \tag{3.10}$$

is a strict supersolution to (2.7) *for* $0 < a < \infty$*;* c *is an arbitrary constant depending only on* $\sup_{n,a} f_n(a,0)$ *in such a way that* (3.5) *holds.*

Proof of Lemma 3.3
We check if $\bar{f}$ is a strict supersolution

$$\begin{aligned}
&\partial_t\bar{f}_n(a,t) + (n-6)\,\partial_a\bar{f}_n(a,t) - \Gamma(f(t))\left(J\bar{f}\right)_n(a,t) \\
&= \bar{f}_n(a,t) - \underbrace{\frac{(n-6)}{n_0}}_{<1}\bar{f}_n(a,t) + \Gamma(f(t))\underbrace{\left(-\left(J\bar{f}\right)_n(a,t)\right)}_{\geq 0} > 0
\end{aligned}$$

and observe that the coupling terms $\left(J\bar{f}\right)_n(a,t)$ were already discussed in Lemma 3.2. As $\bar{f}_n(a,t) > 0$ for $0 < a < \infty$, the proof is done. *q.e.d.*

3.4 Positivity of total number of grains

The goal of this section is to show that the total number of grains $N(t)$ cannot drop down to zero within finite time. We will prove this by contradiction using the conservation of total covered area $A(t)$ and a lemma concerning the tail of $\sum_n f_n(a,t)$ w.r.t. a. We will motivate and derive the key lemma mentioned above now.

According to the *von Neumann–Mullins law* (2.1) incorporated into our system of transport equations (2.7), the highest possible speed by which the area of a grain can shrink is 4. Now consider

$$Q(\sigma,t) = \sum_n \int_\sigma^\infty f_n(a,t)\,da \tag{3.11}$$

and observe that this quantity should be non–decreasing in time if $t \mapsto \sigma(s,t)$ is a characteristic line of the transport operator selected by s with gradient -4 in (a,t)–space.
This observation shall be cast into formulas.

$$\begin{aligned} \frac{d}{dt}\sum_n \int_{\sigma(s,t)}^\infty f_n(a,t)\,da &= \sum_n \int_\sigma^\infty \partial_t f_n(a,t)\,da - \partial_t\sigma \sum_n f_n(\sigma,t) \\ &\overset{*}{=} \sum_n (n-6) f_n(\sigma,t) - \partial_t\sigma \sum_n f_n(\sigma,t) \\ &\geq -\left(4 + \partial_t\sigma(s,t)\right) \sum_n f_n(\sigma(s,t),t) \end{aligned} \tag{3.12}$$

We used (2.7) and (2.9) to obtain the equality $(*)$. The choice of σ as

$$\sigma(s,t) = s - 4t \tag{3.13}$$

is sufficient to ensure the desired inequality $\frac{d}{dt}Q(\sigma,t) \geq 0$. These calculations give rise to the following lemma.

Lemma 3.4 (quantile) *For any given $s_0 \geq 0$ and any finite time $t_0 > 0$ we have*

$$\sum_n \int_{s_0}^\infty f_n(a,t_0+t)\,da \geq \sum_n \int_{s_0+4t}^\infty f_n(a,t_0)\,da \tag{3.14}$$

for any $0 < t < \infty$.

Proof of Lemma 3.4
From the above calculations (3.12) and by plugging in the choice (3.13) of σ we have

$$\sum_n \int\limits_{s-4(t_0+t)}^{\infty} f_n\left(a, t_0+t\right) da \geq \sum_n \int\limits_{s-4t_0}^{\infty} f_n\left(a, t_0\right) da$$

for all times t_0 and t_0+t such that $0 < t_0 < \infty$, $0 < t_0+t < \infty$, and for all $s \geq 4t_0$. Mapping $s \mapsto s+4t$ and labeling $s_0 = s-4t_0$ leads to the desired inequality (3.14). *q.e.d.*

We can exploit Lemma 3.4 to illustrate that grains bigger than a given area 4α "survive" at least for a certain time α. To be more precise, we state the following corollary.

Corollary 3.1 *For any positive, finite times t_1 and t_2 we have*

$$N\left(t_1+t_2\right) \geq \sum_n \int\limits_{4t_2}^{\infty} f_n\left(a, t_1\right) da \tag{3.15}$$

where $N\left(t_1+t_2\right)$ denotes the total number of grains at time t_1+t_2.

Proof of Corollary 3.1
Set $s_0=0$, $t_0=t_1$, and $t=t_2$ in Lemma 3.4. *q.e.d.*

Now we will prove by contradiction to the conservation of total covered area that the total number of grains remains positive within finite times.

Theorem 3.1 *If the initial number of grains is positive and the initial data satisfy the polyhedral formula* (2.13)*, then the number of grains remains positive for all finite times.*

$$N(0) > 0 \quad \Rightarrow \quad N(t) > 0 \quad \forall\, t \in (0, \infty)$$

Proof of Theorem 3.1
Assume there exists a time $0 < t_0 < \infty$ such that $N\left(t_0\right)=0$. Corollary 3.1 to Lemma 3.4 now reads

$$0 = N\left(t_0\right) = \sum_n \int\limits_0^{\infty} f_n\left(a, t_0\right) da \geq \sum_n \int\limits_{4t}^{\infty} f_n\left(a, t_0-t\right) da \geq 0$$

for all $0 < t < t_0$. Mapping $t \mapsto t_0-t$ yields

$$0 = \sum_n \int\limits_{4(t_0-t)}^{\infty} f_n(a, t)\, da$$

for all $0 < t < t_0$. This implies

$$N(t) = \sum_n \int_0^\infty f_n(a,t)\,da = \sum_n \int_0^{4(t_0-t)} f_n(a,t)\,da = \sum_n \int_0^{4t_0} f_n(a,t)\,da$$

for all $0 < t < t_0$. Conservation of total covered area (proved in Lemma 2.4) and again Lemma 3.4 allow for the following estimate

$$\begin{aligned} 1 \equiv A(t) = \sum_n \int_0^\infty a f_n(a,t)\,da &= \sum_n \int_0^{4t_0} a f_n(a,t)\,da \\ &\leq 4t_0 \sum_n \int_0^{4t_0} f_n(a,t)\,da \end{aligned}$$

for all $0 < t < t_0$. So we have

$$N(t) \geq \frac{1}{4t_0}\,, \quad t < t_0.$$

As $N(t)$ is continuous in t we observe

$$0 = \lim_{t \to t_0} N(t) \geq \frac{1}{4t_0} > 0$$

which is contradictory to our assumption on $N(t_0)$. *q.e.d.*

3.5 Bounding total mass from below

Within the previous Section 3.4 we have shown that the total number of grains – or total mass – $N(t)$ cannot drop down to zero within finite times. Unfortunately this result is not uniform in f. It turns out that we can achieve a real a–priori estimate not depending on $f(t)$ in exchange for a dependence on n_0 in the case of initial data with finite support. In case of initial data with infinite support we shall have a closer look on Lemma 3.4 and especially on Corollary 3.1 from the previous Section 3.4 to come up with an a–priori estimate depending on the quantiles of the initial data.

As the denominator of $\Gamma(f(t))$ can be bounded by $N(t)$ from below, the results of this section provide us with the other ingredient (besides the supersolution (3.8) in Lemma 3.2) to bound $\Gamma(f(t))$ itself for finite times.

First we restrict ourselves to initial data with finite support. In that case solutions f to (2.7) have finite support for finite times as well.

Definition 3.2 (length of support) *For a function* $u = (u_2, ..., u_{n_0})$ *we call*

$$\omega(u) := \sup_{n,a} \{a : |u_n(a)| > 0\}$$

the length of the support of u *w.r.t.* a.

Remark 3.3 *The extension of the support of solutions* f *to* (2.7) *is limited by movement along the fastest characteristic lines of the transport operator.*

$$\omega(f(t)) \le \omega(f(0)) + (n_0 - 6)\, t$$

With this knowledge we can easily achieve the desired result for initial data with finite support by exploiting conservation of total covered area.

Lemma 3.5 *If the initial data* g *have finite support and satisfy the polyhedral formula* (2.13) *then the total number of grains of solutions* f *to* (2.7) *can be bounded from below via*

$$N(t) \ge \frac{1}{\omega + (n_0 - 6)\, t} > 0$$

for $0 < t < \infty$ *where* ω *is the length of the support of the initial data.*

Proof of Lemma 3.5
According to Remark 3.3 we have

$$\begin{aligned}\sum_n \int_0^\infty f_n(a,t)\, da &= \sum_n \int_0^{\omega+(n_0-6)t} \frac{a}{a} f_n(a,t)\, da \\ &\ge \frac{1}{\omega + (n_0-6)\, t} \underbrace{\sum_n \int_0^{\omega+(n_0-6)t} a f_n(a,t)\, da}_{=1}\end{aligned}$$

by using conservation of total covered area due to Lemma 2.4. Furthermore we have

$$\frac{1}{\omega + (n_0 - 6)\, t} > 0$$

for all finite $0 \le t < \infty$. *q.e.d.*

Now we focus our attention to initial data with infinite support. The following Lemma is actually – like Corollary 3.1 – a special case of Lemma 3.4 from the previous Section 3.4.

Lemma 3.6 *If the initial data g have infinite support then the total number of grains of solutions f to* (2.7) *can be bounded from below via*

$$N(t) \geq \sum_n \int_{4t}^{\infty} f_n(a,0)\, da > 0$$

where $0 < t < \infty$ is an arbitrary finite time.

Proof of Lemma 3.6
We observe

$$N(t) \geq \sum_n \int_{4t}^{\infty} f_n(a,0)\, da$$

by setting $s_0 = 0$ and $t_0 = 0$ in Lemma 3.4. According to the infinite support of the initial data we have

$$\sum_n \int_{4t}^{\infty} f_n(a,0)\, da > 0$$

for all finite $0 \leq t < \infty$. *q.e.d.*

3.6 Bounding the coupling's weight

We now collect results on bounding the coupling's weight $\Gamma(f(t))$ in terms of the initial data.

Lemma 3.7 *If f is a solution to* (2.7) *and its initial data satisfy the polyhedral formula* (2.13)*, then we can bound $\Gamma(f(t))$ by*

$$\Gamma(f(t)) \leq c_1 (\omega + (n_0 - 6)\, t) \sum_{n=2}^{5} \exp(-\gamma\, n)$$

if the initial data have finite support and by

$$\Gamma(f(t)) \leq c_2 \frac{\sum_{n=2}^{5} \exp(-\gamma\, n)}{\sum_n \int_{4t}^{\infty} f_n(a,0)\, da}$$

if the initial data have infinite support. The constants c_1 and c_2 only depend on $\sup_{n,a} f_n(a,0)$ essentially; ω denotes the length of support of the initial data.

Proof of Lemma 3.7
From Lemma 2.3 we have

$$\Gamma\left(f\left(t\right)\right) \leq c\,\frac{\sum\limits_{n} f_{n}\left(0,t\right)}{\sum\limits_{n}\int\limits_{0}^{\infty} f_{n}\left(a,t\right)da}$$

for any finite $t > 0$. The numerator is now bounded via Lemma 3.2 and the denominator either by Lemma 3.5 or by Lemma 3.6. *q.e.d.*

Chapter 4

Existence of solutions

In this chapter we prove existence of solutions to the infinite–dimensional system (2.3) by taking a suitable limit of solutions to finite–dimensional systems (2.7). Furthermore we use energy methods to prove uniqueness and continuous dependence on the data of solutions to (2.3).

4.1 Finite system

4.1.1 Function spaces and mild solutions

In order to study the existence and other properties of solutions to (2.7), we introduce an open subset X^{n_0} of the Banach function space of integrable and bounded continuous vectorial functions $x = (x_n)_{n=2}^{n_0}$ of dimension $n_0 - 1$ with zero boundary condition for $n > 6$, labelled $\left[L^1 \cap BC_{0,n>6}\right]^{n_0-1}$.

Definition 4.1

$$\begin{aligned} X^{n_0} = & \left\{ x = (x_n)_{n=2}^{n_0} : \mathbb{R}_+ \to \mathbb{R}^{n_0-1} \;\middle|\; \|x\|_1 + \|x\|_\infty + |\Gamma(x)| < \infty \right\} \\ & \cap \left\{ x = (x_n)_{n=2}^{n_0} : \mathbb{R}_+ \to \mathbb{R}^{n_0-1} \;\middle|\; \forall n \,:\, x_n(\cdot) \textit{ continuous} \right\} \\ & \cap \left\{ x = (x_n)_{n=2}^{n_0} : \mathbb{R}_+ \to \mathbb{R}^{n_0-1} \;\middle|\; n > 6 \,:\, x_n(0) = 0 \right\} \end{aligned}$$

$$\|x\|_1 = \sum_n \int_0^\infty |x_n(a)|\, da$$

$$\|x\|_\infty = \sup_{n,a} |x_n(a)|$$

$$|\Gamma(|x|)| = \left| \frac{\sum_n (n-6)^2 |x_n(0)|}{\sum_n n \int_0^\infty |x_n(a)|\, da - 2(\beta+1) \int_0^\infty |x_2(a)|\, da + n_0\beta \int_0^\infty |x_{n_0}(a)|\, da} \right|$$

As the numerator of $\Gamma(f(t))$ depends on $f_n(0,t)$, $2 \leq n \leq 5$, we have to deal with continuous functions in the spatial variable a. We use the norm $\|x\|_1 + \|x\|_\infty$ for elements of the open subset X^{n_0} of the Banach space $\left[L^1 \cap BC_{0,n>6}\right]^{n_0-1}$.

Furthermore we define Y^{n_0} to describe the space of solutions f to (2.7) by considering the supremum w.r.t. time t of elements of X^{n_0}. We use $\sup_t \|f\|_1 + \sup_t \|f\|_\infty$ as natural norm for elements of $C([0,T]\,;\left[L^1 \cap BC_{0,n>6}\right]^{n_0-1})$. Note Y^{n_0} is an open subset of this Banach space.

Definition 4.2

$$
\begin{aligned}
Y^{n_0} = &\left\{ f = (f_n)_{n=2}^{n_0} : [0,T] \to X^{n_0} \;\middle|\; f_n(a,t) \;\; \textit{continuous w.r.t. } t\;\forall\, n, a \right\} \\
&\cap \left\{ f = (f_n)_{n=2}^{n_0} : [0,T] \to X^{n_0} \;\middle|\; \sup_t \|f\|_1 + \sup_t \|f\|_\infty + \sup_t |\Gamma(f)| < \infty \right\}
\end{aligned}
$$

For convenience we introduce the phase space $\mathcal{P}^{n_0}$ describing the initial data.

Definition 4.3 (phase space)

$$
\begin{aligned}
\mathcal{P}^{n_0} = &\left\{ x \in X^{n_0} \;\middle|\; \sum_n \int_0^\infty x_n(a)\,da > 0\,, \sum_n (n-6) \int_0^\infty x_n(a)\,da = 0 \right\} \\
&\cap \left\{ x \in X^{n_0} \mid x_n(a) \geq 0 \;\forall\, n, a \right\}
\end{aligned}
$$

Motivated by Proposition 3.1 in Section 3.1 we introduce the integral operator $\mathcal{I}$ acting on $Y^{n_0} = C^0([0,T]\,; X^{n_0})$.

Definition 4.4

$$
\begin{aligned}
\mathcal{I}(f) &= (\mathcal{I}_n(f))_{n=2}^{n_0} \\
[\mathcal{I}_n(f)](a,t) &= g_n(a-(n-6)\,t) \\
&\quad + \int_0^t \Gamma(f(s))\,(Jf)_n\,(a-(n-6)(t-s)\,,s)\,ds
\end{aligned}
$$

Note we set $f_n(\alpha,t) = 0$ if the argument α is negative in the formulas above.

Our aim is to construct solutions to (2.7) as fixed points of $\mathcal{I}$ on Y^{n_0}. This gives rise to the following definition of a mild solution.

Definition 4.5 (mild solution) *We call a function $f \in Y^{n_0}$ satisfying*

$$
\begin{aligned}
f_n(a,t) &= g_n(a-(n-6)\,t) \\
&\quad + \int_0^t \Gamma(f(s))\,(Jf)_n\,(a-(n-6)(t-s)\,,s)\,ds
\end{aligned}
$$

for all $2 \leq n \leq n_0$*,* $0 < a < \infty$*, and* $t > 0$ *a mild solution to* (2.7)*. We set* $f_n(\alpha, t) = 0$ *if the argument* α *is negative.*

Definition 4.6 (admissible solution) *We call a mild solution* f *with initial data* $g \in \mathcal{P}^{n_0}$ *an admissible solution to* (2.7) *if* f *is an element of* $C^0([0,T]; \mathcal{P}^{n_0})$ *for all finite times.*

4.1.2 Existence for short times

We will construct short–time solutions f in a certain neighbourhood of the initial data g with heavy restrictions on the possible maximal time T.

Definition 4.7 *We consider*

$$Y^{n_0}_{M_{1,\infty,\Gamma}} = \left\{ f \in Y^{n_0} \,\middle|\, \sup_t \|f\|_1 \leq M_1,\ \sup_t \|f\|_\infty \leq M_\infty,\ \sup_t |\Gamma(f)| \leq M_\Gamma \right\}$$

as an appropriate closed subset of Y^{n_0} *to construct mild (and admissible) solutions in the sense of Definitions 4.5 and 4.6.*
M_1*,* M_∞*, and* M_Γ *are positive finite real numbers depending only on the initial data* g *via*

$$M_1 > \|g\|_1, \quad M_\infty > \|g\|_\infty, \quad M_\Gamma > \frac{30 M_\infty}{C_0},$$

where $C_0 > 0$ *is given by*

$$C_0 = (6 - \max\{2(\beta+1), 5\}) \sum_n \int_0^\infty g_n(a)\, da$$

and $\beta \in (0,2)$ *is a free parameter.*

$Y^{n_0}_{M_{1,\infty,\Gamma}}$ is a closed set in $C^0([0,T]; [L^1 \cap BC^0_0]^{n_0-1})$ and we use the natural norm $\sup_t \|f\|_1 + \sup_t \|f\|_\infty$.

For simplicity it is possible to choose the same number for M_1 and M_∞. Furthermore note that $C_0 = (6 - \max\{2(\beta+1), 5\}) \|g\|_1$.

Theorem 4.1 *For given initial data* $g \in \mathcal{P}^{n_0}$ *there exists a unique mild solution* f *in the sense of Definition 4.5 for all times* $t \in [0, T]$ *where* T *is given by*

$$T = \min\{1, T_s, T_c\}$$

$$T_s = \min\left\{\frac{M_1 - \|g\|_1}{M_1 M_\Gamma 4(\beta+1)\,n_0}, \frac{M_\infty - \|g\|_\infty}{M_\infty M_\Gamma 4(\beta+1)\,n_0}, \frac{M_\Gamma C_0 - 30 M_\infty}{M_1 M_\Gamma^2 (\beta+1)\left(4n_0^2 + 32(\beta+1)\right)}\right\}$$

$$\frac{1}{T_c} = \frac{100}{99}\,4(\beta+1)\,n_0(n_0-1)\left[M_\Gamma + (M_1 + M_\infty)\frac{M_\Gamma}{M_\infty}\left(M_\Gamma \frac{\beta+1}{15}\,n_0 + 1\right)\right]$$

and therefore depends continuously on the initial data.

Proof of Theorem 4.1

We intend to apply Banach's fixed point theorem to the operator $\mathcal{I}$ on Y^{n_0}. Therefore we have to check if $\mathcal{I}$ maps $Y^{n_0}_{M_{1,\infty,\Gamma}}$ to itself and if $\mathcal{I}$ is contractive.

The continuity of f w.r.t. a follows from Proposition 4.1 in Subsection 4.1.4. This allows us to evaluate f pointwise in a especially at the boundary $a = 0$ to compute the numerator of $\Gamma(f(t))$.

Now we observe that the boundary conditions

$$f_n(0,t) = 0 \quad \text{for} \quad n > 6$$

are preserved by $\mathcal{I}$ as we set $f_n(a,t) = 0$ for $a < 0$.

$$n > 6: \quad 0 = [\mathcal{I}_n(f)](0,t) = g_n(\underbrace{-(n-6)\,t}_{\leq 0}) + \int_0^t \Gamma(f(s))\,(Jf)_n(\underbrace{-(n-6)(t-s)}_{\leq 0}, s)\,ds$$

The self mapping property of $\mathcal{I}$ on $Y^{n_0}_{M_{1,\infty,\Gamma}}$ can be easily verified for any $T \leq T_s$.

$$\sup_t \|\,[\mathcal{I}(f)](t)\,\|_1 \leq \|g\|_1 + M_1 M_\Gamma 4(\beta+1)\,n_0 T \leq M_1$$
$$\sup_t \|\,[\mathcal{I}(f)](t)\,\|_\infty \leq \|g\|_\infty + M_\infty M_\Gamma 4(\beta+1)\,n_0 T \leq M_\infty$$

Before examining $|\Gamma([\mathcal{I}(f)](t))|$ we carry out an estimate on a part of the denominator of $\Gamma([\mathcal{I}(f)](t))$ depending on the initial data g to recover C_0

$$
\begin{aligned}
&\sum_n n \int_{(6-n)t}^{\infty} g_n(a)\,da - 2(\beta+1)\int_{4t}^{\infty} g_2(a)\,da + n_0\beta\int_0^{\infty} g_{n_0}(a)\,da \\
&\geq 6\sum_n \int_0^{\infty} g_n(a)\,da - \sum_{n=2}^{5} n \int_0^{(6-n)t} g_n(a)\,da - 2(\beta+1)\int_{4t}^{\infty} g_2(a)\,da \\
&\geq 6\sum_n \int_0^{\infty} g_n(a)\,da - 2(\beta+1)\int_0^{\infty} g_2(a)\,da - \sum_{n=3}^{5} n \int_0^{\infty} g_n(a)\,da \\
&\geq\ (6-\max\{2(\beta+1)\,,\,5\})\sum_n \int_0^{\infty} g_n(a)\,da = C_0
\end{aligned}
$$

where $\beta \in (0,2)$. We used the polyhedral formula (2.13) to obtain the first inequality.

So we have

$$
|\Gamma([\mathcal{I}(f)](t))| \leq \frac{30M_\infty}{C_0 - M_1 M_\Gamma(\beta+1)(4n_0^2 + 32(\beta+1))T} \leq M_\Gamma
$$

finally.

Before we start to verify the contraction property w.r.t. $\sup_t\|\cdot\|_1 + \sup_t\|\cdot\|_\infty$ we carry out an auxiliary calculation on $|\Gamma(u)-\Gamma(v)|$ by decomposing $\Gamma(f)$ into it's numerator $\mathcal{N}(f)$ and denominator $\mathcal{D}(f) \geq C_0 > 30M_\infty/M_\Gamma$.

$$
\begin{aligned}
&|\Gamma(u)-\Gamma(v)| = \left|\frac{\mathcal{N}(u)}{\mathcal{D}(u)} - \frac{\mathcal{N}(v)}{\mathcal{D}(v)}\right| \\
\leq& \frac{1}{|\mathcal{D}(u)\mathcal{D}(v)|}\Big(|\mathcal{N}(u)\mathcal{D}(v) - \mathcal{N}(u)\mathcal{D}(u)| + |\mathcal{N}(u)\mathcal{D}(u) - \mathcal{N}(v)\mathcal{D}(u)|\Big) \\
\leq& \frac{M_\Gamma^2}{30M_\infty}|\mathcal{D}(u)-\mathcal{D}(v)| + \frac{M_\Gamma}{30M_\infty}|\mathcal{N}(u)-\mathcal{N}(v)| \\
\leq& \frac{M_\Gamma}{M_\infty}\left(M_\Gamma\frac{\beta+1}{15}n_0\|u-v\|_1 + \|u-v\|_\infty\right)
\end{aligned}
\tag{4.1}
$$

We have

$$\begin{aligned}
&\sup_t \| \left[\mathcal{I}\left(u\right)\right](t) - \left[\mathcal{I}\left(v\right)\right](t) \|_1 \\
&\leq \sup_t 4\left(\beta+1\right) n_0\, t \sum_n \int_0^\infty \sup_s \left|\Gamma\left(u\left(s\right)\right) u_n\left(a,s\right) - \Gamma\left(v\left(s\right)\right) v_n\left(a,s\right)\right| da \\
&\leq 4\,T\left(\beta+1\right) n_0 \sum_n \int_0^\infty \sup_t \left|\Gamma\left(u\left(t\right)\right) u_n\left(a,t\right) - \Gamma\left(v\left(t\right)\right) v_n\left(a,t\right)\right| da \\
&\leq 4\,T\left(\beta+1\right) n_0\left(n_0-1\right) \sup_{n,t} \int_0^\infty \left|\Gamma\left(u\left(t\right)\right) u_n\left(a,t\right) - \Gamma\left(v\left(t\right)\right) v_n\left(a,t\right)\right| da \\
&\leq 4\,T\left(\beta+1\right) n_0\left(n_0-1\right) \sup_t \left(M_\Gamma \| u\left(t\right) - v\left(t\right) \|_1 + M_1 \left|\Gamma\left(u\left(t\right)\right) - \Gamma\left(v\left(t\right)\right)\right|\right)
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\sup_t \| \left[\mathcal{I}\left(u\right)\right](t) - \left[\mathcal{I}\left(v\right)\right](t) \|_\infty \\
&\leq \sup_t 4\left(\beta+1\right) n_0\, t \sup_{n,a} \sup_s \left|\Gamma\left(u\left(s\right)\right) u_n\left(a,s\right) - \Gamma\left(v\left(s\right)\right) v_n\left(a,s\right)\right| \\
&\leq 4\,T\left(\beta+1\right) n_0 \sup_{n,a,t} \left|\Gamma\left(u\left(t\right)\right) u_n\left(a,t\right) - \Gamma\left(v\left(t\right)\right) v_n\left(a,t\right)\right| \\
&\leq 4\,T\left(\beta+1\right) n_0 \sup_{n,a,t} \Big(\left|\Gamma\left(u\left(t\right)\right) u_n\left(a,t\right) - \Gamma\left(u\left(t\right)\right) v_n\left(a,t\right)\right| \\
&\quad + \left|\Gamma\left(u\left(t\right)\right) v_n\left(a,t\right) - \Gamma\left(v\left(t\right)\right) v_n\left(a,t\right)\right| \Big) \\
&\leq 4\,T\left(\beta+1\right) n_0 \sup_t \Big(M_\Gamma \| u\left(t\right) - v\left(t\right) \|_\infty + M_\infty \left|\Gamma\left(u\left(t\right)\right) - \Gamma\left(v\left(t\right)\right)\right| \Big)
\end{aligned} \tag{4.3}$$

for all $T \leq T_c$. We now combine (4.2) with (4.3) and use (4.1) to achieve

$$\begin{aligned}
&\sup_t \| \left[\mathcal{I}\left(u\right)\right](t) - \left[\mathcal{I}\left(v\right)\right](t) \|_1 + \sup_t \| \left[\mathcal{I}\left(u\right)\right](t) - \left[\mathcal{I}\left(v\right)\right](t) \|_\infty \\
&\leq \Big(\sup_t \| u\left(t\right) - v\left(t\right) \|_1 + \sup_t \| u\left(t\right) - v\left(t\right) \|_\infty \Big)\, T \Big[4\left(\beta+1\right) n_0\left(n_0-1\right) \\
&\qquad \Big(M_\Gamma + \left(M_1 + M_\infty\right) \frac{M_\Gamma}{M_\infty} \Big(M_\Gamma \frac{\beta+1}{15} n_0 + 1 \Big) \Big) \Big]
\end{aligned} \tag{4.4}$$

finally.
As $T = \min\{1, T_s, T_c\}$ and Y^{n_0} is a subset of a complete metric space we can apply Banach's fixed point theorem. The unique fixed point of $\mathcal{I}$ on $Y^{n_0}_{M_{1,\infty,\Gamma}}$ is a mild solution to (2.7) in the sense of Definition 4.5.

The continuous dependence of T on the initial data g is clear as the components the minimum is taken from are all continuous in g. *q.e.d.*

4.1.3 Continuous dependence on the data

Lemma 4.1 *The mild solution f in the sense of Definition 4.5 depends continuously on its initial data g.*

Proof of Lemma 4.1
The continuous dependence on the data is a direct consequence of the structure of $\mathcal{I}$ and can be verified by an application of Gronwall's lemma.
Let f^1 and f^2 be mild solutions to the initial data g^1 and g^2. Now we define

$$z(t) := \|f^1 - f^2\|_1 + \|f^1 - f^2\|_\infty$$

and we gain

$$z(t) \le \|g^1 - g^2\|_1 + \|g^1 - g^2\|_\infty + \text{const} \int_0^t z(s)\, ds$$

from a computation similar to the one leading to (4.4) in the proof of Theorem 4.1. f^1 and f^2 are continuous functions in a and t; therefore z is continuous, too. We apply Gronwall's lemma and get

$$z(t) \le \left(\|g^1 - g^2\|_1 + \|g^1 - g^2\|_\infty\right) \exp(\text{const } t)$$

implying the desired continuous dependence on the data. *q.e.d.*

4.1.4 Regularity of mild solutions

Proposition 4.1 *The mild solution f to initial data $g \in \mathcal{P}^{n_0}$ is continuous in a and t.*

Proof of Proposition 4.1
Looking at Definition 4.5 we observe that all components involved are continuous functions: The initial data g are continuous by assumption, $\Gamma(f(t))$ is continuous, the coupling $(Jf)_n(a,t)$ is continuous in a and t if the $f_n(a,t)$ involved are continuous, and the integration mapping is continuous. *q.e.d.*

Corollary 4.1 *If the initial data $g \in \mathcal{P}^{n_0}$ are continuously differentiable, then a mild solution f is continuously differentiable, too.*

Proof of Corollary 4.1
We can construct mild solutions $u = \partial_a f$ to initial data $v = \partial_a g$ as fixed points of $\mathcal{I}$ on Y^{n_0} in the same way as f to initial data g. This is possible as $\Gamma\left(f\left(t\right)\right)$ only depends on $\sum_n \int_0^\infty f_n\left(a,t\right)da$ and $f_n\left(0,t\right)$ such that we have a set of first–order equations in a. We apply Proposition 4.1 to $v = \partial_a g$ and $u = \partial_a f$ to identify a candidate for $\partial_a f$ formally.
Now we consider the difference quotient

$$d_h f_n\left(a,t\right) = \frac{f_n\left(a+h,t\right) - f_n\left(a,t\right)}{h}$$

and use Definition 4.5 to achieve the following bound

$$\|d_h f\left(t\right)\|_\infty \leq \|d_h g\|_\infty + 4\left(\beta+1\right)n_0 \sup_{0\leq s\leq t}\Gamma\left(s\right)\int_0^t \|d_h f\left(s\right)\|_\infty ds$$

on $d_h f_n\left(a,t\right)$. Applying Gronwall's lemma we have

$$\|d_h f\left(t\right)\|_\infty \leq \|d_h g\|_\infty \exp\left(c\left(g,n_0\right)t\right)$$

allowing us to take the limit $h \to 0$. Finally we have to verify $\|d_h f - \partial_a f\|_\infty \to 0$ as $h \to 0$. Again Definition 4.5 implies

$$\begin{aligned}\|d_h f\left(t\right) - u\left(t\right)\|_\infty &\leq \|d_h g - v\|_\infty \\ &\quad + 4\left(\beta+1\right)n_0 \sup_{0\leq s\leq t}\Gamma\left(s\right)\int_0^t \|d_h f\left(s\right) - u\left(s\right)\|_\infty ds\end{aligned}$$

and by Gronwall's lemma we have

$$\|d_h f\left(t\right) - u\left(t\right)\|_\infty \leq \|d_h g - v\|_\infty \exp\left(c\left(g,n_0\right)t\right) \longrightarrow 0\,, \quad h \to 0$$

completing the proof with $u = \partial_a f$ and $v = \partial_a g$. *q.e.d.*

Corollary 4.2 *If the initial data $g \in \mathcal{P}^{n_0}$ are continuously differentiable, then a mild solution f is continuously differentiable in t.*

Proof of Corollary 4.2
We repeat the proof of Corollary 4.1 and apply Proposition 4.1 on $\partial_a g$. Then we differentiate (3.1) w.r.t. t and deduce boundedness of $\partial_t f_n\left(a,t\right)$. *q.e.d.*

Remark 4.1 *Corollaries 4.1 and 4.2 imply that a mild solution to continuously differentiable initial data is a strong solution of* (2.7).

4.1.5 Mild and admissible solutions

Proposition 4.2 *We can approximate mild solutions in Y^{n_0} with initial data in $\mathcal{P}^{n_0}$ by solutions in $Y^{n_0} \cap C^0([0,T]\,;[C^1\,[0,\infty)]^{n_0-1})$ with initial data in $\mathcal{P}^{n_0} \cap [C^1\,[0,\infty)]^{n_0-1}$.*

Proof of Proposition 4.2
The function space C^1 is dense in C^0. The regularity results (Proposition 4.1, Corollary 4.1) on mild solutions f and their continuous dependence on the initial data g (Lemma 4.1) together with the exponentially decaying supersolution in Lemma 3.3 allow for an approximation of mild solutions in Y^{n_0} by mild solutions in $Y^{n_0} \cap C^0([0,T]\,;[C^1]^{n_0-1})$ in the natural norm $\sup_t \|f\|_1 + \sup_t \|f\|_\infty$. *q.e.d.*

Proposition 4.3 *We can approximate mild solutions in Y^{n_0} with initial data in $\mathcal{P}^{n_0}$ by solutions in $Y^{n_0} \cap C^1([0,T]\,;[C^0\,[0,\infty)]^{n_0-1})$ with initial data in $\mathcal{P}^{n_0} \cap [C^1\,[0,\infty)]^{n_0-1}$.*

Proof of Proposition 4.3
Proposition 4.2 states that we can approximate mild solutions in Y^{n_0} by mild solutions in $Y^{n_0} \cap C^0([0,T]\,;[C^1\,[0,\infty)]^{n_0-1})$. Furthermore (2.7) reads as

$$\partial_t f_n\,(a,t) = -\,(n-6)\,\partial_a f_n\,(a,t) + \Gamma\,(f\,(t))\,(Jf)_n\,(a,t)$$

and due to Lemma 4.1 the r.h.s. of this equation is continuous w.r.t. t. This implies that f is in $Y^{n_0} \cap C^1([0,T]\,;[C^0]^{n_0-1})$. *q.e.d.*

Propositions 4.2 and 4.3 allow us to extend the a–priori calculations from Chapter 2 and Chapter 3 to mild solutions in Y^{n_0}.

Corollary 4.3 *The mild solution with initial data $g \in \mathcal{P}^{n_0}$ is an admissible solution for all times $t \in [0,T\,]$, i.e. $f \in C^0\,([0,T]\,;\mathcal{P}^{n_0})$.*

Proof of Corollary 4.3
We approximate the solution in Y^{n_0} by solutions in

$$Y^{n_0} \cap C^0([0,T]\,;\left[C^1\right]^{n_0-1}) \cap C^1([0,T]\,;\left[C^0\right]^{n_0-1})\,.$$

Due to Lemma 3.1 in Section 3.2 we have $f \geq 0$. Furthermore we obtain from Lemma 2.2 in Subsection 2.3.2 that f fulfills the polyhedral formula (2.13) for all $t \in (0,T\,]$. Finally Lemmas 3.5 and 3.6 imply positivity of $\sum \int f_n\,(a,t)\,da > 0$. *q.e.d.*

4.1.6 Existence for arbitrary finite times

Theorem 4.2 *For given initial data $g \in \mathcal{P}^{n_0}$ there exists a unique admissible solution $f \in C^0([0,T];\mathcal{P}^{n_0})$ where $T < \infty$ is arbitrary.*

Proof of Theorem 4.2
The strategy of the proof is to divide $(0,T]$ into appropriate subintervals $(T_{j-1},T_j]$, $0 = T_0 < T_1 < \cdots < T_{k-1} < T_k = T$, and to apply Theorem 4.1 on each of these subintervals. Therefore we have to ensure that the constants M_1, M_∞, and M_Γ do not need to be changed throughout the whole sequence of subintervals $((T_{j-1},T_j])_j$.

- M_1 is defined to be strictly bigger than $\|g\|_1$ and as the total number of grains decreases in time (Corollary 2.1) we have $\|g\|_1 \geq \|f\|_1$ for all times.
- M_∞ is defined to be strictly bigger than $\|g\|_\infty$ and due to the supersolution in Lemma 3.2, which is constant in a, we have $\|g\|_\infty \geq \|f\|_\infty$ for all times by using a comparison principle.
- M_Γ depends on M_∞ (which remains the same) and the number of grains $N(T_{j-1})$ at the beginning of each subinterval.

  In the case of finite support of the initial data we have

  $$N(t) \geq \frac{1}{\omega + (n_0 - 6)\,t} > 0 \qquad \text{(Lemma 3.5)}$$

  where ω denotes the length of the support of the initial data.

  If the initial data have infinite support we know

  $$N(t) \geq \sum_n \int_{4t}^{\infty} f_n(a,0)\,da > 0 \qquad \text{(Lemma 3.6)}$$

  for all finite times t. It is sufficient to choose M_Γ as $M_\Gamma(T)$.

Therefore the reapplication of Theorem 4.1 and Corollary 4.3 on each of the subintervals $(T_{j-1},T_j]$ is allowed.
Proposition 4.1 states the continuity of f. *q.e.d.*

4.2 Infinite system

The goal of this section is to construct mild solutions of (2.3) by taking a limit of admissible solutions of the finite–dimensional system (2.7). Further properties of solutions to (2.3) will be studied in Chapter 5.

4.2.1 Appropriate spaces and mild solutions

We now introduce the Banach sequence space X as the infinite–dimensional analogue to X^{n_0} (cf. Definition 4.1). Again the natural norm used is $\|x\|_1 + \|x\|_\infty$.

Definition 4.8

$$\begin{aligned} X = &\left\{x = (x_n)_{n=2}^{\infty},\, x_n : \mathbb{R}_+ \to \mathbb{R} \mid \|x\|_1 + \|x\|_\infty + |\Gamma(x)| < \infty\right\} \\ &\cap \left\{x = (x_n)_{n=2}^{\infty},\, x_n : \mathbb{R}_+ \to \mathbb{R} \mid \forall n : x_n(\cdot) \text{ continuous}\right\} \\ &\cap \left\{x = (x_n)_{n=2}^{\infty},\, x_n : \mathbb{R}_+ \to \mathbb{R} \mid n > 6 : x_n(0) = 0\right\} \end{aligned}$$

$$\|x\|_1 = \sum_{n=2}^{\infty} \int_0^{\infty} |x_n(a)|\, da$$

$$\|x\|_\infty = \sup_{n,a} |x_n(a)|$$

$$|\Gamma(|x|)| = \left| \frac{\sum_{n=2}^{\infty} (n-6)^2 |x_n(0)|}{\sum_{n=2}^{\infty} n \int_0^{\infty} |x_n(a)|\, da - 2(\beta+1) \int_0^{\infty} |x_2(a)|\, da} \right|$$

Furthermore we define Y to describe the space of solutions f to (2.3) by considering the supremum w.r.t. time t of elements of X using the natural norm $\sup_t \|f\|_1 + \sup_t \|f\|_\infty$.

Definition 4.9

$$\begin{aligned} Y = &\left\{f = (f_n)_{n=2}^{\infty} : [0,T] \to X \mid f_n(a,t) \text{ continuous w.r.t. } t\, \forall n, a\right\} \\ &\cap \left\{f = (f_n)_{n=2}^{\infty} : [0,T] \to X \;\middle|\; \sup_t \|f\|_1 + \sup_t \|f\|_\infty + \sup_t |\Gamma(f)| < \infty\right\} \end{aligned}$$

Remark 4.2 *We consider elements of X^{n_0} as elements of X (and elements of Y^{n_0} as elements of Y) by setting $x_n = 0$ (and $y_n = 0$) for $n > n_0$.*

We extend our definition of mild solutions to (2.7) (cf. Definition 4.5) to the infinite–dimensional case.

Definition 4.10 *We call a function $f \in Y$ satisfying*

$$\begin{aligned} f_n(a,t) = &g_n(a - (n-6)t) \\ &+ \int_0^t \Gamma(f(s)) (Jf)_n (a - (n-6)(t-s), s)\, ds \end{aligned}$$

for all $n \geq 2$, $0 < a < \infty$, and $t > 0$ a mild solution to (2.3). *We set $f_n(\alpha, t) = 0$ if the argument α is negative.*

We will prove existence (cf. Subsection 4.2.2) and uniqueness (cf. Subsection 4.2.3) of solutions to (2.3) in a subset $Y^{(1)} \subset Y$ where all component functions f_n are continuously differentiable w.r.t. a and t.

Definition 4.11

$$Y^{(1)} = \left\{ f \in Y \;\middle|\; \forall n \geq 2 \,:\, f_n \in C^0\left([0,T]\,; C^1[0,\infty)\right) \cap C^1\left([0,T]\,; C^0[0,\infty)\right) \right\}$$

4.2.2 Existence for finite times

Theorem 4.3 *For given continuously differentiable initial data $g \geq 0$ in X which are finite w.r.t. n (i.e. $\exists\, n_0 < \infty : \sup_a g_n(a) = 0 \;\forall n > n_0$) and satisfy the polyhedral formula (2.13) there exists a solution $f \geq 0$ in $Y^{(1)}$ for all times $t \in [0,T]$ where $T < \infty$ is arbitrary.*
Furthermore f satisfies the polyhedral formula and the total covered area $A(t)$ is conserved.

Proof of Theorem 4.3
The main idea of the proof is to construct mild solutions to (2.3) as a limit of admissible solutions to (2.7) where the largest possible $n = n_0$ is increasing. We label the final n, namely n_0, by k from now on to keep notations simple. Furthermore we introduce the following notation

$$\left(\left(f_n^k\right)_{n=2}^{k}\right)_{k>6}$$

for a sequence $\left(f^k\right)_k$ of solutions $f^k = \left(f_n^k\right)_{n=2}^{k}$ to finite–dimensional systems. Note that elements f^k of the sequence $\left(f^k\right)_k$ only make sense for $k > 6$ due to (2.13). Before we start to identify weak limits of this sequence in Y we carry out some calculations bounding the spatial and the time derivatives of the $f_n^k(a,t)$ by the initial data g. As these bounds will be independent of k we omit the upper index k and write $f_n(a,t)$ instead.

Due to Lemma 3.2 we have a supersolution

$$\bar{f}_n(a,t) = \frac{c}{\beta n} \exp(-\gamma n)\,, \quad 2 \leq n \leq k$$

for f^k independent of k. It is very easy to compute that we also have a subsolution

$$\underline{f}_n(a,t) = -\bar{f}_n(a,t) < 0\,, \quad 2 \leq n \leq k$$

of the same structure.
Note that the coupling's weight $\Gamma(f(t))$ does not depend on a directly but only on $\sum f_n(0,t)$ and $\sum \int f_n(a,t)\,da$. Therefore the spatial derivative of a solution satisfies an equation analogue to (2.7) where $\Gamma = \Gamma(t)$ is known. Hence we can bound the spatial derivative $\partial_a f_n^k$ by the supersolution $\bar{f}_n(a,t)$ and the subsolution $-\bar{f}_n(a,t)$ depending only on the spatial derivative $\partial_a g$ of the initial data.

We can also bound the time derivative of solutions to (2.7) via

$$\begin{aligned}
&|\partial_t f_n(a,t)| \\
&\quad \leq |(n-6)\,\partial_a f_n(a,t)| + 4(\beta+1)\sup_t \Gamma(f(t))\,(n+1) \sup_{n-1\leq l\leq n+1} f_l(a,t) \\
&\quad \leq \left(c_2\,|n-6| + 4(\beta+1)\,c_1\,(n+1)\sup_t \Gamma(f(t))\right)\frac{\exp(-\gamma(n-1))}{\beta(n-1)} \\
&\quad \leq c_0 \frac{n+6}{n-1}\exp(-\gamma n)
\end{aligned}$$

due to the previous considerations and Lemma 3.7 (bounding $\sup_t \Gamma$). The constants c_1 and c_2 only depend on the initial data and their spatial derivative, c_0 also depends on T (via $\sum_n \int_{4T}^{\infty} g_n(a)\,da$).

We have that $f_n(a,t)$, $|\partial_a f_n(a,t)|$, and $|\partial_t f_n(a,t)|$ are uniformly bounded by $\sup_{n\geq 2} c\,\frac{n+6}{n-1}\exp(-\gamma n)$ on $(0,\infty)\times(0,T]$ for all $n \geq 2$ (independent of a and t). Therefore $\left(f^k\right)_k$ is equicontinuous.

Now consider an arbitrary, but fixed n. By Arzela–Ascoli we know that there exists a subsequence $(f_n^{k_\nu})_{k_\nu}$ such that $f_n^{k_\nu} \to f_n$ uniformly on a compact subset of $(0,\infty)\times(0,T]$ as $\nu \to \infty$. Furthermore f_n is continuous w.r.t. both variables a and t.
We exhaust $(0,\infty)\times(0,T]$ (and especially $(0,\infty)$) with compact subsets. On each of these subsets we apply the above argument, i.e. there exists a subsequence $(f_n^{k_\alpha})_{k_\alpha}$ such that $f_n^{k_\alpha} \to f_n$ pointwise where α denotes an indexing of the compact subsets (e.g. with length increment 1).
By choosing an appropriate diagonal subsequence

$$\left(f_n^{k_{\mu(\nu,\alpha)}}\right)_{k_\mu}$$

we deduce that $f_n^{k_\mu} \to f_n$ pointwise, f_n continuous, on $(0,\infty)\times(0,T]$ as $\mu \to \infty$. Note $f_n^{k_\mu} \equiv 0$ for $n > k_\mu$.

We start executing the procedure described above for $n = 2$ and achieve that $f_2^{k_\mu} \to f_2$ pointwise, f_2 continuous, on $(0,\infty)\times(0,T]$.

The sequence $(f_3^{k_\mu})_{k_\mu}$ is also bounded and equicontinuous. This implies – again by Arzela–Ascoli – that there exists a further subsequence $(f_3^{k_\lambda})_{k_\lambda}$ which converges pointwise to a continuous f_3. This convergence is again uniform on compact subsets of $(0,\infty)\times(0,T]$.

Proceeding as above implies that we can choose a suitable diagonal subsequence $(f^{k_\kappa})_{k_\kappa}$ such that $f_n^{k_\kappa}\longrightarrow f_n$ pointwise, f_n continuous, for all n as $\kappa\longrightarrow\infty$.

Now we observe that our bounds on $\Gamma\left(f^{k_\kappa}(t)\right)$ (cf. Lemmas 2.3 and 3.7) are independent of k_κ as we omit the term $k_\kappa\beta\int_0^\infty f_{k_\kappa}^{k_\kappa}(a,t)\,da$ while estimating the denominator of Γ.
It remains to show that $\Gamma\left(f^{k_\kappa}(t)\right)\to\Gamma(f(t))$ pointwise in time in the limit $\kappa\longrightarrow\infty$:
The convergence of the numerator is clear as it contains only weighted contributions $f_n(0,t)$ for $n=2,\ldots,5$ due to the boundary conditions (2.6).
A–priori calculations stated in Lemma 5.3 (Chapter 5) imply convergence of the denominator's essential part $\sum_n n\int_0^\infty f_n(a,t)\,da$ and its last term $n_0\beta\int_0^\infty f_{n_0}(a,t)\,da$ in the limit process $f^{k_\kappa}\longrightarrow f$ as $\kappa\longrightarrow\infty$.
Convergence of the term $-2(\beta+1)\int_0^\infty f_2(a,t)\,da$ is trivial. Therefore the limit $\Gamma(f(t))$ is given by (2.5).

With this knowledge we are able to pass to the limit in the integral formulation

$$\begin{aligned} f_n^{k_\kappa}(a,t) = {} & g_n(a-(n-6)\,t) \\ & + \int_0^t \Gamma\left(f^{k_\kappa}(s)\right)\left(Jf^{k_\kappa}\right)_n(a-(n-6)(t-s),s)\,ds \end{aligned} \tag{4.5}$$

and achieve a mild solution to (2.3) in the sense of Definition 4.10. This solution is non–negative and continuous in time (and space).

We observe that $\|f(t)\|_1 = N(t)\le N(0)$ (cf. Lemma 5.5), $\|f(t)\|_\infty\le C(g)$ (cf. Lemma 3.2), and $\Gamma(f(t))\le c(g)$ (cf. Lemma 3.7). Therefore the limit sequence $f=\lim_{\kappa\to\infty}f^{k_\kappa}$ is still bounded w.r.t. the corresponding norms.

In order to prove differentiability of solutions w.r.t. a and t we differentiate the integral formulation (4.5) and bound the modulus of the r.h.s. independently of k_κ. We first investigate

$$\begin{aligned}
&\left|\int_0^t \Gamma\left(f^{k_\kappa}(s)\right)\left(J\partial_a f^{k_\kappa}\right)_n(a-(n-6)(t-s),s)\,ds\right| \\
&\leq C_\Gamma(f(\cdot,0),t)\int_0^t \left|\left(J\partial_a f^{k_\kappa}\right)_n(a-(n-6)(t-s),s)\right| ds \\
&\leq C_\Gamma(f(\cdot,0),t)\,\frac{3}{2}\,4(\beta+1)\exp(\gamma)\,\frac{C_0(f(\cdot,0))}{\beta}\,t\,\exp(-\gamma n)
\end{aligned}$$

using Lemma 3.7 to bound Γ and the supersolution (and subsolution) to control the derivatives within the integrand.
Therefore differentiating (4.5) w.r.t. a leads to

$$\left|\partial_a f_n^{k_\kappa}(a,t)\right| \leq \left|\partial_a g_n(a-(n-6)t)\right| + c(f(\cdot,0),t)\;t\;\exp(-\gamma n)$$

and we can pass to the limit $\kappa\to\infty$ as the initial data are smooth. Differentiating (4.5) w.r.t. t gives us

$$\begin{aligned}
\left|\partial_t f_n^{k_\kappa}(a,t)\right| \leq{}& |n-6|\;\left|\partial_a g_n(a-(n-6)t)\right| \\
&+c(f(\cdot,0),t)\;t\;|n-6|\;\exp(-\gamma n) \\
&+\underbrace{\left|\Gamma\left(f^{k_\kappa}(t)\right)\left(Jf^{k_\kappa}\right)_n(a,t)\right|}_{\leq C(f(\cdot,0),t)\;\exp(-\gamma n)}
\end{aligned}$$

using the Leibniz integral rule. Again we can pass to the limit $\kappa\to\infty$ as the initial data are smooth and finite w.r.t. n.

Conservation of the polyhedral formula is stated in Lemma 5.4 under the assumptions of this theorem.

Corollary 5.1 in Section 5.2 implies that total covered area is conserved as we can plug $(f^{k_\kappa})_{k_\kappa}$ into the proof of Lemma 5.2, set $\alpha = k_\kappa$, and pass to the limit (w.r.t. $\kappa\to\infty$). Note that the appearing sums are cut off at k_κ by definition of f^{k_κ}. *q.e.d.*

4.2.3 Energy methods and uniqueness

Within this subsection we will show that solutions to (2.3) in the sense of Theorem 4.3 are unique and depend continuously on the initial data. The proof is carried out using energy methods.
We start our considerations with a weighted L^2–energy aiming at getting (almost) a sign for the coupling terms.

Definition 4.12 *We define*

$$E_s(t) = \sum_n n \int_0^\infty (f_n(a,t))^2 \, da$$

for solutions f to (2.3)*.*

Lemma 4.2 *For $0 < t \leq T < \infty$ we have*

$$E_s(t) \leq \exp(c\,t)\, E_s(0)$$

for solutions in the sense of Theorem 4.3 where $c = c(f(\cdot,0),T)$ is a constant.

Proof of Lemma 4.2
First we investigate how the coupling terms behave when they are multiplied by nf_n. These computations are inspired by an integration by parts with a continuous variable n but carried out purely discrete by index shifts and using binomial formulas (cf. Appendix A).

$$\begin{aligned}
&\sum_n n f_n (Jf)_n \\
&\quad = \beta \left(\sum_{n\geq 2} (n+1) f_{n+1} n f_n - n^2 f_n^2 + \sum_{n>2} (n-1) f_{n-1} n f_n - n^2 f_n^2 \right) \\
&\qquad + \sum_{n>2} (n+1) f_{n+1} n f_n - n^2 f_n^2 + 3 f_3 2 f_2 \\
&\quad = -\left(\beta + \frac{1}{2}\right) \sum_{n\geq 2} ((n+1) f_{n+1} - n f_n)^2 + 2 f_2^2
\end{aligned} \tag{4.6}$$

The proof is carried out by differentiating the energy w.r.t. t and using (2.3).

$$\begin{aligned}
&\frac{d}{dt} E_s(t) = 2 \sum_n n \int_0^\infty f_n(a,t)\, \partial_t f_n(a,t)\, da \\
&\quad = \sum_n n(n-6)(f_n(0,t))^2 + 2\,\Gamma(f(t)) \sum_n n \int_0^\infty f_n(a,t)\,(Jf)_n(a,t)\, da \\
&\quad \overset{(4.6)}{\leq} 4\,\Gamma(f(t)) \int_0^\infty (f_2(a,t))^2\, da \\
&\quad \leq c(f(\cdot,0),T) \sum_n n \int_0^\infty (f_n(a,t))^2\, da
\end{aligned}$$

Note that $f_n(0,t) = 0$ for $n > 6$ due to the boundary conditions (2.6) and remind Lemma 3.7 to bound $\Gamma(f(t))$. *q.e.d.*

We can also establish a linear growth rate of $E_s(t)$.

Lemma 4.3 *For $0 < t \leq T < \infty$ we have*

$$E_s(t) \leq C\,t + E_s(0)$$

for solutions in the sense of Theorem 4.3 where $C = C(f(\cdot,0))$ is a constant.

Proof of Lemma 4.3
We repeat our calculations within the proof of Lemma 4.2, decompose Γ into it's numerator and denominator, and use Hölder's inequality to proceed.

$$\begin{aligned}\frac{d}{dt}E_s(t) \leq \cdots \leq 4\,\Gamma(f(t)) \int\limits_0^\infty (f_2(a,t))^2\,da \\ \leq 4\,\tilde{c}\sum_{n=2}^{5}(n-6)^2 f_n(0,t)\,\sup_a f_2(a,t)\,\frac{\int_0^\infty f_2(a,t)\,da}{\sum_n \int_0^\infty f_n(a,t)\,da} \\ \leq C(f(\cdot,0))\end{aligned}$$

The last inequality is obtained by using the supersolution (3.8). *q.e.d.*

Considering the difference of two solutions it turns out that $E_s(t)$ is insufficient to get a result as in Lemma 4.2. Therefore we introduce an "energy" $E(t)$ with a slightly modified integrand and an additional term.

Definition 4.13 *We define*

$$E(t) = \sum_n n \int\limits_0^\infty \exp(-a)\,(f_n(a,t))^2\,da + (N(t))^2$$

for solutions f to (2.3) where $N(t) = \sum \int f_n(a,t)\,da$.

We could also use $\exp(-k\,a)$ instead of $\exp(-a)$ for any $k > 0$ within the definition of $E(t)$ and still acquire the same results that are presented below.

Remark 4.3 *For $E(t)$ we can achieve the same results (Lemmas 4.2 and 4.3) as for $E_s(t)$. The only technical differences are an additional integration by parts w.r.t. a and the use of Cauchy's inequality with ε in order to absorb $(\dot{N})^2$ into the negative boundary term of the partial integration mentioned before.*

From now on we regard the difference of two solutions u and v as the argument of $E(t)$ instead of f.

Lemma 4.4 *For $0 < t \leq T < \infty$ we consider*

$$E(u-v)(t) = \sum_n n \int_0^\infty \exp(-a)\left(u_n(a,t) - v_n(a,t)\right)^2 da + \left(N(u(t)) - N(v(t))\right)^2$$

for any two solutions u and v to (2.3) *in the sense of Theorem 4.3. Then we have*

$$E(u-v)(t) \leq \exp(ct)\, E(u-v)(0)$$

where $c = c(u(\cdot,0), v(\cdot,0), T)$ is a constant.

Proof of Lemma 4.4
Again will carry out the proof by differentiating $E(u-v)(t)$ and using (2.3). For simplicity we omit the arguments (a,t) of u and v wherever possible.

$$\begin{aligned}\frac{d}{dt}E(u-v)(t) &= 2\sum_n n(6-n)\int_0^\infty \exp(-a)(u_n - v_n)(\partial_a u_n - \partial_a v_n)\, da \\ &+ 2\sum_n n \int_0^\infty \exp(-a)(u_n - v_n)\left(\Gamma(u)(Ju)_n - \Gamma(v)(Jv)_n\right) da \\ &+ 2\Big(N(u) - N(v)\Big)\Big(\dot{N}(u) - \dot{N}(v)\Big)\end{aligned} \tag{4.7}$$

The first term on the r.h.s. of (4.7) is treated via an integration by parts:

$$\begin{aligned}&2\sum_n n(6-n)\int_0^\infty \exp(-a)\frac{1}{2}\partial_a(u_n - v_n)^2\, da \\ &= -\sum_n n(6-n)\left(u_n(0,t) - v_n(0,t)\right)^2 \\ &\quad + \sum_n n(6-n)\int_0^\infty \exp(-a)(u_n - v_n)^2\, da \\ &\leq 4\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)^2\, da - \sum_{n=2}^{5} n(6-n)\left(u_n(0,t) - v_n(0,t)\right)^2\end{aligned} \tag{4.8}$$

The second term on the r.h.s. of (4.7) has to be split into several parts in order to overcome the nonlinearity $\Gamma = \mathcal{N}/\mathcal{D}$ (which we also split into it's numerator $\mathcal{N}$ and denominator $\mathcal{D}$.

$$\begin{aligned}
&2\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)\left(\Gamma(u)(Ju)_n - \Gamma(v)(Jv)_n\right)da \\
&= 2\Gamma(u)\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)\left((Ju)_n - (Jv)_n\right)da \\
&+ 2\frac{1}{\mathcal{D}(u)}\left(\mathcal{N}(u) - \mathcal{N}(v)\right)\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)(Jv)_n\,da \\
&- 2\frac{\mathcal{N}(v)}{\mathcal{D}(u)\mathcal{D}(v)}\left(\mathcal{D}(u) - \mathcal{D}(v)\right)\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)(Jv)_n\,da
\end{aligned} \tag{4.9}$$

We now treat the first term on the r.h.s. of (4.9)

$$\begin{aligned}
&2\Gamma(u)\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)\left((Ju)_n - (Jv)_n\right)da \\
&\leq 4\,C_1(u(\cdot,0),T)\int_0^\infty \exp(-a)(u_2 - v_2)^2\,da
\end{aligned}$$

by a computation similar to (4.6) within the proof of Lemma 4.2.
The second term on the r.h.s. of (4.9) is estimated by using Young's inequality and (4.10):

$$\begin{aligned}
&2\frac{1}{\mathcal{D}(u)}\left(\mathcal{N}(u) - \mathcal{N}(v)\right)\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)(Jv)_n\,da \\
&\leq \varepsilon\left(\mathcal{N}(u) - \mathcal{N}(v)\right)^2 + \frac{1}{\varepsilon\mathcal{D}^2(u)}\left(\sum_n n\int_0^\infty \exp(-a)\left|u_n - v_n\right|\,\left|(Jv)_n\right|da\right)^2 \\
&\leq \varepsilon\, c(u(\cdot,0), v(\cdot,0))\sum_{n=2}^{5}\left(u_n(0,t) - v_n(0,t)\right)^2 \\
&\quad + \frac{1}{\varepsilon}\tilde{c}(u(\cdot,0), v(\cdot,0), T)\sum_n n\int_0^\infty \exp(-a)(u_n - v_n)^2\,da
\end{aligned}$$

Here ε has to be chosen again in such a way that the first term of the last inequality above is less than half of the modulus of the second term of the last inequality within (4.8).

The third term on the r.h.s. of (4.9) can be treated by using Cauchy's inequality and again (4.10):

$$
\begin{aligned}
&-2\,\frac{\mathcal{N}(v)}{\mathcal{D}(u)\,\mathcal{D}(v)}\left(\mathcal{D}(u)-\mathcal{D}(v)\right)\sum_n n\int_0^\infty \exp(-a)\,(u_n-v_n)\,(Jv)_n\,da\\
&\le C\left(u(\cdot,0),v(\cdot,0),T\right)\left(\mathcal{D}(u)-\mathcal{D}(v)\right)^2\\
&\qquad + C\left(u(\cdot,0),v(\cdot,0),T\right)\left(\sum_n n\int_0^\infty \exp(-a)\,|u_n-v_n|\;|(Jv)_n|\,da\right)^2\\
&\le \left(6+2(\beta+1)\right)^2 C\left(N(u)-N(v)\right)^2\\
&\qquad + \tilde{c}\left(v(\cdot,0)\right) C\sum_n n\int_0^\infty \exp(-a)\,(u_n-v_n)^2\,da
\end{aligned}
$$

Recall that N is a notation for the total number of grains in contrast to $\mathcal{N}$ which is the numerator of Γ.

Now we present the computation (4.10) that was used above:

$$
\begin{aligned}
&\left(\sum_n n\int_0^\infty \exp(-a)\,|u_n-v_n|\;|(Jv)_n|\,da\right)^2\\
&\le \left(6(\beta+1)\sum_n n\int_0^\infty \exp(-a)\,|u_n-v_n|\,da\, n\sup_{n-1\le k\le n+1}\sup_a v_k\right)^2\\
&\le c\left(v(\cdot,0)\right)\left(\sum_n n\exp(-\gamma n)\int_0^\infty \exp(-a)\,|u_n-v_n|\,da\right)^2\\
&\le c\left(v(\cdot,0)\right)\sum_n n\exp(-\gamma n)\sum_n n\exp(-\gamma n)\left(\int_0^\infty \exp(-a)\,|u_n-v_n|\,da\right)^2\\
&\le \tilde{c}\left(v(\cdot,0)\right)\sum_n n\int_0^\infty \exp(-a)\,(u_n-v_n)^2\,da
\end{aligned}
\tag{4.10}
$$

The estimates (4.10) are mainly achieved by using the supersolution (3.8) and Hölder's inequality twice.

The third term on the r.h.s. of (4.7) can be handled by Cauchy's inequality with ε and by using the supersolution to estimate the intermediate terms of $(\dot{N}(u)-\dot{N}(v))^2$. Here ε has to be chosen in such a way that the second term of the last inequality within (4.11) is less than half of the modulus of the second term of the last inequality within (4.8).

$$
\begin{aligned}
& 2\Big(N\left(u\right)-N\left(v\right)\Big)\Big(\dot{N}\left(u\right)-\dot{N}\left(v\right)\Big) \\
\leq\ & \frac{1}{2\varepsilon}\Big(N\left(u\right)-N\left(v\right)\Big)^2+\varepsilon\Big(\dot{N}\left(u\right)-\dot{N}\left(v\right)\Big)^2 \\
\leq\ & \frac{1}{2\varepsilon}\Big(N\left(u\right)-N\left(v\right)\Big)^2+\varepsilon\, c\left(u\left(\cdot,0\right),v\left(\cdot,0\right)\right)\sum_{n=2}^{5}\left(u_n\left(0,t\right)-v_n\left(0,t\right)\right)^2
\end{aligned}
\tag{4.11}
$$

Combining (4.8), (4.9), (4.11) and the estimates above we can estimate (4.7) by

$$
\frac{d}{dt}E\left(u-v\right)\left(t\right)\leq c\left(u\left(\cdot,0\right),v\left(\cdot,0\right),T\right)E\left(u-v\right)\left(t\right)
$$

for all $0<t\leq T<\infty$ implying the result immediately. *q.e.d.*

Lemma 4.4 now implies the results on uniqueness and continuous dependence on the data of strong solutions to (2.3) we were aiming at.

Corollary 4.4 *Solutions (to non–trivial initial data, i.e. with mass not only in f_6) provided by Theorem 4.3 are unique and depend continuously on their initial data.*

Proof of Corollary 4.4
Uniqueness follows directly from Lemma 4.4 by setting $u\left(\cdot,0\right)=v\left(\cdot,0\right)$. Continuous dependence of solutions on the initial data is also a direct consequence of Lemma 4.4. *q.e.d.*

Chapter 5

Properties of solutions to the infinite system

In Section 4.2 of the previous chapter we have proven existence of strong solutions to (2.3). Now we focus on verifying that properties which hold for the finite system (2.7) (e.g. conservation of total covered area) also to be true in the infinite–dimensional case.

5.1 No runoff at infinity

Within this section we carry out some a–priori calculations concerning the infinite system (2.3). From Lemma 3.2 we know that the amount of mass in $f_n(a,t)$ is exponentially small w.r.t. n. Now we will show that the amount of mass in $f_n(a,t)$ is also exponentially small with increasing a, which means that there is no runoff at infinity.

Definition 5.1 *For any given $\nu > 6$ and $\alpha > 0$ we define*

$$N^{\perp}(t) = N^{\perp}(t,\alpha,\nu) = \sum_{n=\nu+1}^{\infty} \int_0^{\alpha} f_n(a,t)\, da + \sum_{n=2}^{\infty} \int_{\alpha}^{\infty} f_n(a,t)\, da$$

as the non–essential part or "quasi–complement" of $N(t)$.

The idea is now to choose $\nu(t)$ and $\alpha(t)$ as monotonically increasing functions of time to control $N^{\perp}(t)$. A first step is to understand how $N^{\perp}$ evolves in time if ν and α grow.

Lemma 5.1 *Suppose f is a solution to* (2.3)*. Let $\mu(t) > 6$ and $\alpha(t) > 0$ be monotonically increasing functions with $\frac{d}{dt}\alpha(t) \geq \mu - 6$. Then we have*

$$N^{\perp}(t, \alpha(t), \lfloor \mu(t) \rfloor) \leq N^{\perp}(0, \alpha(0), \lfloor \mu(0) \rfloor) + c \int_0^t \exp(-\gamma \mu(s))\, \alpha(s)\, ds \tag{5.1}$$

for all finite times t; c is an arbitrary constant depending on $\sup_{n,a} f_n(a, 0)$ *(in such a way that* (3.5) *holds) and on* $\sup_{0 \leq t \leq T} \Gamma(f(t))$*. Furthermore $\gamma = \log(1 + \frac{1}{\beta})$ is a constant, too.*

Proof of Lemma 5.1
We define

$$\nu(t) = \lfloor \mu(t) \rfloor$$

as the integer part of $\mu(t)$ and denote the jump of $\nu(t)$ by $[\![\nu]\!]$.
We first consider the case $[\![\nu]\!] = 0$. Differentiating $N^{\perp}(t)$ and using (2.3) yields

$$\begin{aligned}
\frac{d}{dt} N^{\perp}(t) &= \frac{d}{dt}\left(\sum_{n=\nu+1}^{\infty} \int_0^{\alpha} f_n(a,t)\, da + \sum_{n=2}^{\infty} \int_{\alpha}^{\infty} f_n(a,t)\, da \right) \\
&= -\sum_{n=\nu+1}^{\infty} (n-6) f_n(\alpha, t) + \Gamma(f(t)) \sum_{n=\nu+1}^{\infty} \int_0^{\alpha} (Jf)_n(a,t)\, da \\
&\quad + \sum_{n=\nu+1}^{\infty} \dot{\alpha} f_n(\alpha, t) \\
&\quad - \sum_{n=2}^{\infty} \int_{\alpha}^{\infty} (n-6)\, \partial_a f_n(a,t)\, da + \Gamma(f(t)) \int_{\alpha}^{\infty} \underbrace{\sum_{n=2}^{\infty} (Jf)_n(a,t)}_{=0}\, da \\
&\quad - \sum_{n=2}^{\infty} \dot{\alpha} f_n(\alpha, t)
\end{aligned} \tag{5.2}$$

using (2.9). Furthermore we observe

$$\sum_{n=\nu+1}^{\infty} (Jf)_n(a,t) = \beta \nu f_{\nu}(a,t) - (\beta+1)(\nu+1) f_{\nu+1}(a,t)$$

as a consequence of (2.4). We estimate the r.h.s. of the above equation (5.2)

$$\begin{aligned}\frac{d}{dt}N^{\perp}(t) &\leq \sum_{n=2}^{\nu}\underbrace{((n-6)-\dot{\alpha})}_{\leq 0} f_n(\alpha,t) + \Gamma(f(t))\int_0^{\alpha}\beta\nu f_\nu(a,t)\,da \\ &\leq \sup_{0\leq t\leq T}\Gamma(f(t))\,c_0\exp(-\gamma\nu(t))\,\alpha(t) \\ &\leq c\,\exp(-\gamma\mu(t))\,\alpha(t)\end{aligned} \tag{5.3}$$

by elementary calculations and Lemma 3.2, c_0 depending only on the initial data by (3.5). Due to Lemma 3.2 and Lemma 3.6, $\sup_{0\leq t\leq T}\Gamma(f(t))$ is a–priori known for finite times T. We also absorb a factor $\exp(\gamma)$ into the constant c besides $\sup_{0\leq t\leq T}\Gamma(f(t))$ and c_0.
In the case $[\![\nu]\!] \neq 0$ we observe that

$$[\![\nu(t)]\!] = \lfloor\mu(t_+)\rfloor - \lfloor\mu(t_-)\rfloor = 1$$

and so we have

$$\begin{aligned}[\![N^{\perp}(t)]\!] &= N^{\perp}(t_+) - N^{\perp}(t_-) \\ &= -\int_0^{\alpha} f_{\lfloor\mu(t_-)\rfloor}(a,t)\,da \leq 0\end{aligned}$$

as an additional part in the r.h.s. of (5.2).
As $[\![N^{\perp}(t)]\!]$ is non–positive, we can come up with the same estimate (5.3) as in the case $[\![\nu]\!] = 0$. *q.e.d.*

Remark 5.1 *We can assume $\frac{d}{dt}\alpha(t) \geq \mu$ instead of $\frac{d}{dt}\alpha(t) \geq \mu - 6$ within Lemma 5.1 to keep notation as simple as possible in the following.*

In order to exploit Lemma 5.1 we have to choose the functions $\mu(t)$ and $\alpha(t)$ depending on each other via $\frac{d}{dt}\alpha(t) \geq \mu(t)$ in a clever way; α should grow at least linearly in ν to compensate for the transport in a along characteristic lines. On the other hand ν should grow exponentially controlling the diffusion in n.

For simplicity we use the notation

$$N^{\perp}(t,\alpha) := N^{\perp}(t,\alpha,\lfloor\alpha\rfloor)$$

if the third argument of $N^{\perp}$ is given by rounding down the second one.

Theorem 5.1 *Consider the functions* $\alpha(t) = \alpha_0 \exp(t)$ *and* $\mu(\alpha(t)) = \alpha(t)$*. Furthermore denote by* $\nu(t) = \lfloor \mu(\alpha(t)) \rfloor$ *the integer part of* μ*. If* f *is a solution to* (2.3) *we have*

$$N^{\perp}(t,\alpha) \leq N^{\perp}(0, \alpha \exp(-t)) + \frac{c}{\gamma} \exp(-\gamma \alpha \exp(-t))$$

for all finite times $t > 0$ *and all finite* $\alpha_0 > 0$*.*

Proof of Theorem 5.1
Instead of (5.1) we consider an equation for the supersolution $\overline{N^{\perp}}$ of $N^{\perp}$, namely

$$\overline{N^{\perp}}(t,\alpha) = N^{\perp}(0,\alpha_0) + c \int_0^t \exp(-\gamma \mu(s))\, \alpha(s)\, ds$$

for any arbitrary $\alpha_0 > 0$.
Now we plug in our ansatz for α and μ:

$$\overline{N^{\perp}}(t,\alpha) = N^{\perp}(0,\alpha_0) + c \int_0^t \exp(-\gamma \alpha_0 \exp(s))\, \alpha_0 \exp(s)\, ds$$

Changing variables $a = \alpha_0 \exp(s)$ leads to

$$\overline{N^{\perp}}(t,\alpha) = N^{\perp}(0,\alpha_0) + c \int_{\alpha_0}^{\alpha_0 \exp(t)} \exp(-\gamma a)\, da$$

and by lengthening the domain of integration (and using $\alpha_0 = \alpha \exp(-t)$) we have

$$\overline{N^{\perp}}(t,\alpha) \leq N^{\perp}(0,\alpha \exp(-t)) + c \int_{\alpha \exp(-t)}^{\infty} \exp(-\gamma a)\, da \tag{5.4}$$

finally. Carrying out the integration completes the proof. *q.e.d.*

Theorem 5.1 implies exponential control of $N^{\perp}$ w.r.t. a in the following way: If we vary the starting point α_0 of an "effective" characteristic line $\alpha(t)$ by δ, then – under the assumptions of Theorem 5.1 – it's endpoint α_T at time T varies by $\delta \exp(T)$. This is still a finite excess as long as T is finite.
The loss of smallness of $N^{\perp}$ by following $\alpha(t)$ is unimportant as we can move the starting point α_0 close to the end of effective support of f at time t_0 and be sure that α_T moves close to the end of effective support of f at time T – only delayed by a finite factor $\exp(T)$.

5.2 Conservation of total covered area

The purpose of this section is to show that total covered area $A(t)$ of solutions f to (2.3) is a conserved quantity for suitable initial data g.

Lemma 5.2 *For a solution f of (2.3) we have*

$$\sum_{n=2}^{\infty}\int_{\alpha}^{\infty} af_n(a,t)\,da \le \alpha N^{\perp}(0,\alpha\exp(-T)) + \int_{\alpha}^{\infty} N^{\perp}(0,a\exp(-T))\,da + c_0(1+\alpha)\exp(-\gamma\alpha\exp(-T))$$

for all times t with $0 < t < T < \infty$. The constant c_0 depends only on T, $\beta \in (0,2)$, and $sup_{n,a} f_n(a,0)$.

Proof of Lemma 5.2
We have

$$\sum_{n=2}^{\infty}\int_{\alpha}^{\infty} af_n(a,t)\,da = \sum_{n=2}^{\infty}\int_{\alpha}^{\infty}\int_{a}^{\infty} f_n(s,t)\,ds\,da + \alpha\sum_{n=2}^{\infty}\int_{\alpha}^{\infty} f_n(a,t)\,da$$

via an integration by parts. Theorem 5.1 implies

$$\alpha\sum_{n=2}^{\infty}\int_{\alpha}^{\infty} f_n(a,t)\,da \le \alpha\left(N^{\perp}(0,\alpha\exp(-t)) + \frac{c}{\gamma}\exp(-\gamma\alpha\exp(-t))\right)$$

and

$$\sum_{n=2}^{\infty}\int_{\alpha}^{\infty}\int_{a}^{\infty} f_n(s,t)\,ds\,da \le \int_{\alpha}^{\infty} N^{\perp}(0,a\exp(-t))\,da + \exp(t)\frac{c}{\gamma^2}\exp(-\gamma\alpha\exp(-t))$$

for any finite time $t > 0$. *q.e.d.*

We recall the definition of $A(t)$ (cf. Definition 2.2) in the infinite–dimensional case

$$A(t) = \sum_{n=2}^{\infty}\int_{0}^{\infty} af_n(a,t)\,da$$

and prove the announced assertion.

Corollary 5.1 *If the initial data g decay exponentially w.r.t. n and a, total covered area $A(t)$ is conserved by solutions f of (2.3) for any finite time $t > 0$.*

Proof of Corollary 5.1
Pass to the limit $\alpha \to \infty$ in Lemma 5.2. Observe $\sum_{n=2}^{\infty}\int_0^{\omega} af_n(a,t)\,da \to 0$ as $\omega \to 0$ due to Lemma 3.2. *q.e.d.*

5.3 Validity of triple junction condition

Within this section we want to argue that the polyhedral formula (cf. Subsection 2.3.2, Proposition 2.2) is preserved in the infinite–dimensional case, too. A first step is to understand why a key quantity within the computations for the finite–dimensional case – $\sum_n n \int_0^\infty f_n(a,t)\,da$ – is also bounded for solutions of (2.3).

Lemma 5.3 *For a solution f of (2.3), we have*

$$\sum_{n=\nu}^{\infty} n \int_0^\infty f_n(a,t)\,da \le \nu N^{\perp}(0, \nu \exp(-T)) + \int_\nu^\infty N^{\perp}(0, a\exp(-T))\,da$$
$$+ c_0 (1+\nu) \exp(-\gamma \nu \exp(-T))$$

for all times t with $0 < t < T < \infty$. The constant c_0 depends on T, $\beta \in (0,2)$, and $sup_{n,a} f_n(a,0)$.

Proof of Lemma 5.3
We split the integration at n

$$\sum_{n=\nu}^{\infty} n \int_0^\infty f_n(a,t)\,da = \sum_{n=\nu}^{\infty} n \int_0^n f_n(a,t)\,da + \sum_{n=\nu}^{\infty} n \int_n^\infty f_n(a,t)\,da$$
$$\le \frac{c}{\beta} \sum_{n=\nu}^{\infty} n \exp(-\gamma n) + \sum_{n=\nu}^{\infty} \int_n^\infty a f_n(a,t)\,da$$

and bound the first term according to Lemma 3.2. The second term is estimated by using the increasing weight $a \ge n$ within the integrals. Furthermore we have

$$\sum_{n=\nu}^{\infty} \int_n^\infty a f_n(a,t)\,da \le \sum_{n=2}^{\infty} \int_\nu^\infty a f_n(a,t)\,da$$

which can be bounded via Lemma 5.2. Now we estimate the series

$$\sum_{n=\nu}^{\infty} n \exp(-\gamma n) \le c_1 (1+\gamma\nu) \exp(-\gamma\nu) \le c_2 (1+\nu) \exp(-\gamma\nu\exp(-T))$$

and combine this with the result of Lemma 5.2. *q.e.d.*

Now we can state the conservation of the polyhedral formula in the infinite–dimensional case.

Lemma 5.4 *If the initial data g are continuously differentiable and satisfy*

$$\sum_{n=2}^{\infty}(n-6)\int_{0}^{\infty}g_n(a)\,da=0$$

then the polyhedral formula for solutions f of (2.3)

$$\sum_{n=2}^{\infty}(n-6)\int_{0}^{\infty}f_n(a,t)\,da=0$$

holds for all times $0<t\le T<\infty$.

Proof of Lemma 5.4
We consider a suitable diagonal sequence (f^{k_κ}) of solutions to the finite–dimensional system (2.7) as in the proof of Theorem 4.3. For each f^{k_κ} we can carry out the proof of Lemma 2.2. Bounding $\left|\partial_t f_n^{k_\kappa}(a,t)\right|$ in the same way as within the proof of Theorem 4.3 allows us to differentiate w.r.t. t.

$$\begin{aligned}
&\frac{d}{dt}\sum_{n=2}^{k_\kappa}(n-6)\int_{0}^{k_\kappa}f_n^{k_\kappa}(a,t)\,da=\sum_{n=2}^{k_\kappa}(n-6)\int_{0}^{k_\kappa}\partial_t f_n^{k_\kappa}(a,t)\,da\\
=&\sum_{n=2}^{k_\kappa}(n-6)\int_{0}^{k_\kappa}\Gamma\left(f^{k_\kappa}(t)\right)\left(Jf^{k_\kappa}\right)_n(a,t)\,da-\sum_{n=2}^{k_\kappa}(n-6)^2\int_{0}^{k_\kappa}\partial_a f_n^{k_\kappa}(a,t)\,da\\
=&\Gamma\left(f^{k_\kappa}(t)\right)\sum_{n=2}^{k_\kappa}(n-6)\int_{0}^{k_\kappa}\left(Jf^{k_\kappa}\right)_n(a,t)\,da\\
&+\sum_{n=2}^{k_\kappa}(n-6)^2\left(f_n^{k_\kappa}(0,t)-f_n^{k_\kappa}(k_\kappa,t)\right)
\end{aligned}$$

Theorem 5.1 and especially Lemma 5.3 ensure that all terms in the above calculations remain bounded while passing to the limit $\kappa\to\infty$. Using the zero balance property (2.9) and some index shifts gives us

$$\begin{aligned}
\sum_{n=2}^{k_\kappa}n\left(Jf^{k_\kappa}\right)_n&=\beta\sum_{n=3}^{k_\kappa}n(n-1)f_{n-1}^{k_\kappa}+(\beta+1)\sum_{n=2}^{k_\kappa-1}n(n+1)f_{n+1}^{k_\kappa}\\
&\quad-\beta\sum_{n=2}^{k_\kappa-1}n^2f_n^{k_\kappa}-(\beta+1)\sum_{n=3}^{k_\kappa}n^2f_n^{k_\kappa}\\
&=-\left(\sum_{n=2}^{k_\kappa}nf_n^{k_\kappa}-2(\beta+1)f_2^{k_\kappa}+k_\kappa\beta f_{k_\kappa}^{k_\kappa}\right)
\end{aligned}$$

for the weighted sum of coupling terms. Passing to the limit $\kappa \to \infty$ leads to the desired result

$$\begin{aligned}\frac{d}{dt}\sum_{n=2}^{\infty}(n-6)\int_0^{\infty} f_n(a,t)\,da \\ &= \sum_{n=2}^{\infty}(n-6)^2 f_n(0,t) \\ &\quad - \Gamma(f(t))\left(\sum_n n\int_0^{\infty} f_n(a,t)\,da - 2(\beta+1)\int_0^{\infty} f_2(a,t)\,da\right) \\ &= 0\end{aligned}$$

by the choice of $\Gamma(f(t))$ in (2.5) and the *zero balance property* (2.9) of the coupling, i.e. $\sum_n (Jf)_n(a,t) = 0$. Note that

$$k_\kappa \int_0^\infty f_{k_\kappa}^{k_\kappa}(a,t)\,da = k_\kappa\left(\int_0^{k_\kappa} f_{k_\kappa}^{k_\kappa}(a,t)\,da + \int_{k_\kappa}^\infty f_{k_\kappa}^{k_\kappa}(a,t)\,da\right) \to 0$$

as $\kappa \to \infty$ due to Lemma 3.2 and Lemma 5.3. Furthermore we observe $\sum_{n=2}^{k_\kappa}(n-6)^2 f_n^{k_\kappa}(k_\kappa,t) \to 0$ as $\kappa \to \infty$ due to Theorem 5.1 and again Lemma 5.3 as $f_n^{k_\kappa}(a,t)$ is continuous w.r.t. a. *q.e.d.*

This result supports the choice of $\Gamma(f(t))$ in (2.5) as a consequence of the calculations in the proof of Lemma 2.2.

5.4 Decrease of total number of grains

The last fact which we can expect to verify is that total number of grains (cf. Subsection 2.3.1, Definition 2.1)

$$N(t) = \sum_{n=2}^{\infty}\int_0^{\infty} f_n(a,t)\,da$$

is a decreasing quantity in the infinite–dimensional case, too.

Lemma 5.5

$$\frac{d}{dt}N(t) = \sum_{n=2}^{5}(n-6)f_n(0,t) \le 0$$

Proof of Lemma 5.5
Theorem 5.1 implies that the boundary values at $a = 0$ for $n = 2, \ldots, 5$ are

the only "drainage" (for suitable initial data) – no mass can be lost at infinity w.r.t. a or n. Therefore the proof is essentially the same as in Lemma 2.1 and Corollary 2.1.

We consider a suitable diagonal sequence (f^{k_κ}) of solutions to the finite–dimensional system (2.7) as in the proof of Theorem 4.3. For each f^{k_κ} we can carry out the proof of Lemma 2.1, but we integrate only up to k_κ instead of ∞ (w.r.t. a). We also sum up to k_κ only.
Bounding $\left|\partial_t f_n^{k_\kappa}(a,t)\right|$ in the same way as within the proof of Theorem 4.3 allows us to differentiate w.r.t. t. We use the zero balance property (2.9) and pass to the limit $\kappa \to \infty$ with $f_n^{k_\kappa}(a,t)$. Convergence of the appearing terms is ensured by Theorem 5.1. *q.e.d.*

Chapter 6

Long–time behaviour

We intend to investigate some features of the long–time behaviour of solutions to (2.3). Besides a characterization of stationary solutions we will focus our attention on self–similar solutions.

6.1 Stationary solutions

Lemma 6.1 (characterization of stationary solutions) *Nontrivial stationary solutions to* (2.3)*, i.e.* $f^s \not\equiv 0$*, can be characterized by*

$$f_n^s(a) = 0 \qquad n \neq 6$$

for all $0 < a < \infty$*.*

Remark 6.1 *The component function* $f_6^s(a)$ *can be anything.*

Proof of Lemma 6.1
The criterion for a stationary solution is

$$-(n-6)\,\partial_a f_n^s(a) + \Gamma(f^s)\,(Jf^s)_n(a) = 0$$

for all $n \geq 2$ and $0 < a < \infty$.
Summing over n and integrating w.r.t. a yields

$$0 = -\sum_n (n-6) \int_0^\infty \partial_a f_n^s(a)\, da = \sum_n (n-6)\, f_n^s(0)$$

by using the zero balance property (2.9). Having a closer look on the outcome of the above calculation we observe

$$f_n^s(0) = 0 \qquad \text{for } 2 \leq n \leq 5 \tag{6.1}$$

as $f_n^s(0) = 0$ for $n > 6$ due to the boundary conditions (2.6) and the contribution from the term $n = 6$ is zero due to the prefactor $(n-6)$. Furthermore we have $f_n^s(0) \geq 0$ for all $2 \leq n \leq n_0$. This implies

$$\Gamma(f^s) = 0$$

due to the definition of Γ in (2.5). Now the criterion for stationary solutions reduces to

$$(n-6)\,\partial_a f_n^s(a) = 0$$

for all $n \geq 2$ and $0 < a < \infty$. Together with (6.1) and the boundary conditions (2.6) this completes the proof. *q.e.d.*

Lemma 6.2 *A nontrivial stationary solution as described in Lemma 6.1 is not attractive as slightly perturbed data lead to a positive* $\Gamma(f(t))$ *for some times* t *and are therefore affected by the coupling operator* $(Jf)_n(a,t)$.

Proof of Lemma 6.2
Assume there exists a finite time t such that we have $\int f_6(a,t)\,da = 1-\varepsilon$, $\sum_{n>6}(n-6)\int f_n(a,t)\,da = \varepsilon/2$, and $\Gamma(f(t)) = 0$. Then (2.13) implies $\sum_{n<6}(6-n)\int f_n(a,t)\,da = \varepsilon/2$. This gives us an upper bound on the time τ after which $\Gamma(f(t))$ is positive: $\tau = \min_a\{f_n(a,t^*) > 0,\ 2 \leq n \leq 5\}$. At that time two things happen: 1^{st} the total number of grains decreases, and 2^{nd} the discrete diffusion is active and shuffles mass away from $\int f_6$. Then we can repeat our estimate on τ. *q.e.d.*

We are not able to prove that the trivial stationary solution is attractive for nonstationary initial data, i.e. $\exists k \geq 2,\ k \neq 6, : \int g_k(a)\,da > 0$, but we show that the total number of grains $N(t)$ is strictly decreasing for most finite times t.

Lemma 6.3 *The total number of grains* $N(t)$ *to nonstationary initial data, i.e.* $\exists k \geq 2,\ k \neq 6, : \int g_k(a)\,da > 0$, *decreases strictly for most finite times.*

Proof of Lemma 6.3
Lemma 2.1 implies that the total number of grains $N(t)$ decreases in time. Due to (2.13) and the leftwards transport for $n < 6$ we observe that $\dot{N} < 0$ for most times. We will elaborate on this. Assume there is a time t^* such that $f_n(0,t) = 0$ for all n at $t = t^*$. The polyhedral formula (2.13) implies $\sum_{n=2}^{5}\int f_n(a,t)\,da > 0$. We have an upper bound on the time τ after which at least one $f_n(0,t)$, $2 \leq n \leq 5$, is strictly positive implying $\dot{N} < 0$ at least at that time $\tau = \min_a\{f_n(a,t^*) > 0,\ 2 \leq n \leq 5\}$. *q.e.d.*

6.2 Self–similar scaling

Self–similar scaling behaviour, also called normal grain growth, often occurs in experiments. To observe this in our model we start our analysis by rescaling equations (2.3) (cf. Subsection 2.2.1) and considering stationary solutions of the rescaled system. The resulting system of ordinary differential equations will be examined afterwards.

6.2.1 Natural rescaling

Following an ansatz by Fradkov [5] we introduce the relative quantity

$$\xi = \frac{a}{M} = a\,N$$

to replace the spatial variable a. Here $M = M\left(f\left(t\right)\right)$ denotes the mean grain area (cf. Subsection 2.3.3, Definition 2.3). Lemma 2.4 implies that M is the inverse of the total number of grains N. In order to retain total covered area A as a conserved quantity we use the following scaling

$$f_n\left(a,t\right) = N^2 \varphi_n\left(\xi,t\right)$$

for $n \geq 2$ and $a \in (0, \infty)$. This scaling is consistent with $A \equiv 1$

$$\sum_n \int_0^\infty \xi \varphi_n\left(\xi,t\right) d\xi = N^{-2} \sum_n \int_0^\infty aN f_n\left(a,t\right) N da \equiv 1$$

and provides a second conserved quantity

$$\sum_n \int_0^\infty \varphi_n\left(\xi,t\right) d\xi = N^{-2} \sum_n \int_0^\infty f_n\left(a,t\right) N da \equiv 1$$

for the rescaled functions $\varphi_n\left(\xi,t\right)$ in the new variable ξ. Furthermore we still have

$$\sum_n \left(n-6\right) \int_0^\infty \varphi_n\left(\xi,t\right) d\xi = N^{-1} \sum_n \left(n-6\right) \int_0^\infty f_n\left(a,t\right) da = 0$$

which is important to compute the rescaled $\Gamma\left(f\left(t\right)\right)$. We have

$$\Gamma\left(f\right) = \frac{\sum \left(n-6\right)^2 f_n\left(0,t\right)}{\sum n \int f_n\, da - 2\left(\beta+1\right) \int f_2\, da} = N \frac{\sum \left(n-6\right)^2 \varphi_n\left(0\right)}{6 - 2\left(\beta+1\right) \int \varphi_2\, da} =: N \mathcal{G}\left(\varphi\right)$$

and

$$(Jf)_n(a,t) = N^2 (J\varphi)_n(\xi,t)$$

for the terms on the r.h.s. of (2.3). The derivatives on the l.h.s. change via

$$(n-6)\,\partial_a f_n(a,t) = (n-6)\,N^3 \partial_\xi \varphi_n(\xi,t)$$

and

$$\partial_t f_n(a,t) = 2N\dot{N}\varphi_n(\xi,t) + N\dot{N}\xi\partial_\xi\varphi_n(\xi,t) + N^2\partial_t\varphi_n(\xi,t)$$

by using $aN = \xi$ within the second term of the time derivative. Collecting terms and dividing by N^3 leads to

$$N^{-1}\partial_t\varphi_n + \left(n - 6 + \frac{\dot{N}}{N^2}\xi\right)\partial_\xi\varphi_n = \mathcal{G}(\varphi)\,(J\varphi)_n - 2\frac{\dot{N}}{N^2}\varphi_n \qquad (6.2)$$

as the rescaled version of (2.3). Using Lemma 2.1 (or better Lemma 5.5) we observe

$$\dot{N}(t) = \sum_n (n-6)\,f_n(0,t) = N^2 \sum_n (n-6)\,\varphi_n(0,t) =: -N^2\alpha(t) \qquad (6.3)$$

and by considering stationary solutions of (6.2) and plugging in our definition of α (6.3) we immediately get

$$(n - 6 - \alpha\xi)\,\partial_\xi\varphi_n = \mathcal{G}(\varphi)\,(J\varphi)_n + 2\alpha\varphi_n \qquad (6.4)$$

with positive boundary conditions for $\varphi_n(0)$ where $2 \leq n \leq 5$ and zero boundary conditions elsewhere. Note that the factor α in (6.4) can be scaled out mainly by using a different $G(\varphi)$ instead of $\mathcal{G}(\varphi)$.

6.2.2 Simple rescaling

We can also rescale (2.3) directly by

$$\varphi_n(\xi,t) = t^2 f_n(a,t)\ , \quad \xi = \frac{a}{t}$$

and consider stationary solutions to achieve a simpler form of (6.4):

$$(n - 6 - \xi)\,\partial_\xi\varphi_n = G(\varphi)\,(J\varphi)_n + 2\varphi_n\,, \quad n \geq 2 \qquad (6.5)$$

These equations are also subject to the boundary conditions

$$\varphi_n(0) = 0 \qquad (6.6)$$

for $n > 6$. The nonlinearity takes the following form:

$$G(\varphi) = \frac{\sum (n-6)^2 \varphi_n(0)}{6 \sum \int \varphi_n \, d\xi - 2(\beta+1) \int \varphi_2 \, d\xi} \tag{6.7}$$

This simple scaling also ensures conservation of total covered area (before considering stationary solutions)

$$\sum_n \int_0^\infty \xi \varphi_n(\xi, t) \, d\xi = t^2 \sum_n \int_0^\infty a \, t^{-1} f_n(a, t) \, t^{-1} da \equiv 1$$

and furthermore we can recover some information on $\sum \int \varphi_n \, d\xi$ by integrating (6.5) and summing up:

$$-\sum_n (n-6) \varphi_n(0) = \sum_n \int_0^\infty \varphi_n(\xi) \, d\xi \tag{6.8}$$

Here we used the zero balance property (2.9) of the coupling $(J\varphi)_n$.

6.2.3 Formal solution

We can solve any equation of the system (6.5) at least formally by considering the homogeneous equation

$$(n-6-\xi) \, \partial_\xi \varphi_n + b_n \varphi_n = 0$$

where

$$\begin{aligned} b_n &= n \, G(2\beta+1) - 2 \\ b_2 &= 2 \, G\beta - 2 \end{aligned}$$

for $n \geq 2$ first. This equation can be solved by separation of variables and logarithmic integration. We incorporate a r.h.s. $G(K\varphi)_n$ consisting of the $(n+1)$ and $(n-1)$ parts of the coupling $G(J\varphi)_n$ using a variation of constants formula and integrating φ_{n+1} and φ_{n-1}. For that purpose we introduce the notation $(K\varphi)_n$ denoting the parts of the coupling $(J\varphi)_n$

$$\begin{aligned} (K\varphi)_n &= (\beta+1)(n+1) \, \varphi_{n+1} + \beta(n-1) \, \varphi_{n-1} \\ (K\varphi)_2 &= (\beta+1) \, 3\varphi_3 \end{aligned}$$

not containing φ_n itself. Considering $2 \leq n \leq 6$ we get

$$\varphi_n(\xi) = (\xi+6-n)^{b_n} \, G \int_\xi^\infty (x+6-n)^{-(b_n+1)} \, (K\varphi)_n(x) \, dx \tag{6.9}$$

for $\xi \in (0, \infty)$ using the natural behaviour at infinity $\lim_{\xi\to\infty} \varphi_n(\xi) = 0$. Taking the initial condition (6.6) $\varphi_n(0) = 0$ for $n > 6$ into account we achieve

$$\varphi_n(\xi) = (n-6-\xi)^{b_n} G \int_0^{\xi} (n-6-x)^{-(b_n+1)} (K\varphi)_n(x)\, dx \tag{6.10}$$

for $\xi \in (0, n-6)$. Again assuming $\lim_{\xi\to\infty} \varphi_n(\xi) = 0$ for $n > 6$ we have

$$\varphi_n(\xi) = (\xi+6-n)^{b_n} G \int_{\xi}^{\infty} (x+6-n)^{-(b_n+1)} (K\varphi)_n(x)\, dx \tag{6.11}$$

for $\xi \in (n-6, \infty)$.

We do not prove existence of solutions to (6.5) as two major difficulties arise: First there are no a–priori bounds available (like supersolutions) as in the time–dependent case and second the zeros of $n-6-\xi$ in front of the derivatives $\partial_\xi \varphi_n(\xi)$ are not that easy to overcome due to the alternating influence of the φ_n by the coupling $(J\varphi)_n$.

Good news is that we can bound the nonlinearity $G(\varphi)$ from above and below uniformly

$$\frac{1}{6} \leq G \leq \frac{12}{6-2(\beta+1)} \tag{6.12}$$

using (6.7) and (6.8).

The remaining task is to select a self–similar solution that is “physically reasonable”, i.e. covers the same total area A as the solution to (2.3) in the time–dependent case. We observe that $\sum \int \xi\varphi_n(\xi)\, d\xi$ depends continuously on the weighted sum of the initial values $\sum_{n=2}^{5} (6-n)\, \alpha_n$.

Remark 6.2 *It is unclear if we can reach $\sum_{n=2}^{\infty} \int_0^{\infty} \xi\, \varphi_n(\xi)\, d\xi = A$ for any given finite A by a choice of finite initial values α_n, $2 \leq n \leq 5$.*

To achieve consistency with experimental results [7] we can demand

$$\int_0^{\infty} \varphi_2(\xi)\, d\xi \ll \sum_{n\geq 2} \int_0^{\infty} \varphi_n(\xi)\, d\xi$$

as an additional selection criterion, e.g. by prescribing $\int \varphi_2(\xi)\, d\xi = \varepsilon N$ for any given $0 < \varepsilon \ll 1$ where ε should be extracted from experimental data.

6.3 Topological classes distribution

Assuming solutions to (6.5) exist, we can integrate the equations w.r.t. ξ and obtain a system of equations relating the topological class distributions $\phi_n = \int \varphi_n(\xi)\, d\xi$:

$$(6-n)\,\varphi_n(0) = G(\varphi)\,(J\phi)_n + \phi_n\,, \quad n \geq 2 \tag{6.13}$$

Note that $\varphi_n(0) = 0$ for $n \geq 6$. Given $\phi_2, \varphi_2(0), \ldots, \varphi_5(0)$, and β we can use (6.13) to compute all ϕ_n, $n > 2$, successively. $G(\varphi)$ is fully determined by these input parameters due to (6.7) and (6.8). These parameters, especially the initial values $\varphi_n(0)$, $2 \leq n \leq 5$, have to be selected such that the relations (6.8), (2.13), and $\sum \int \varphi_n(\xi)\, d\xi = A$ hold.

We are able to compute the topological classes distribution for suitable sets of input parameters. The initial values $\varphi_2(0), \ldots, \varphi_5(0)$ are chosen such that $\sum (n-6) \int \varphi_n(\xi)\, d\xi = 0$ and $\sum (6-n)\, \varphi_n(0) = \sum \int \varphi_n(\xi)\, d\xi$ hold and also all ϕ_n are non–negative. This choice is not unique and we can also adjust the ratios of the ϕ_n, $n < 6$, a bit in this way. $\beta = 1/2$ is chosen according to [8] and in common with [7] we set $\phi_2 = 10^{-3}$ as starting value for the iteration. The condition $\sum \int \xi\varphi_n(\xi)\, d\xi = A$ is satisfied by a suitable rescaling of the $\varphi_n(0)$. The iteration is carried out up to $n = 1000$. Besides a typical shape

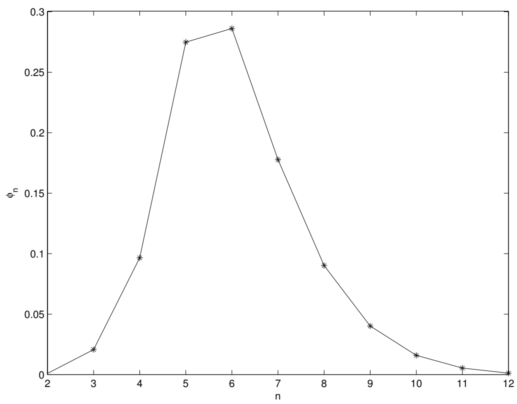

Figure 6.1: Topological classes distribution

of the topological classes distribution (cf. Figure 6.1) we observe exponential decay of the ϕ_n for increasing n. The results are in common with Fradkov et al. [8, Fig. 1] and also with simulation results of a completely different model [9, Fig. 9] we treated earlier. The parameter β seems to influence the ratio between ϕ_5 and ϕ_6 and also the decay rate of the ϕ_n.

6.4 Lewis' law

A natural question concerning grain growth is to ask whether there is a correlation between the topological class and the area of a grain. Lewis observed a linear relationship [13] examining cellular structures arising in biology. In common with Flyvbjerg [4] this so–called *Lewis' law* reads

$$\langle \xi \rangle_n = a\,(n-6) + b \tag{6.14}$$

concerning our model (in scale invariant, dimensionless variables). Here $\langle \xi \rangle_n = \int \xi \varphi_n(\xi)\, d\xi / \int \varphi_n(\xi)\, d\xi$ denotes the mean value of $\varphi_n(\xi)$. It is unclear if this phenomenological law is really applicable for grain growth. Rivier and Lissowski derived *Lewis' law* by maximum entropy arguments applied to cell distributions [18].

Concerning our model, arguments given by Flyvbjerg are reasonable that we expect *Lewis' law* as a consequence of *von Neumann–Mullins law*, as we describe the fundamental dynamics of grain growth through one of it's consequences [4], namely the *von Neumann–Mullins law* (2.1).
Furthermore we observe that *Lewis' law* cannot be true for small n and arbitrary a, $b \geq 0$. In the sequel we will show that *Lewis' law* is valid for asymptotically large n. Similar results are achieved by Flyvbjerg [4].

Proposition 6.1 *Assume there exists a smooth solution to* (6.5)*. Then we have the asymptotics*

$$\phi_n = \left(\frac{\beta}{\beta+1}\right)^n n^{\frac{1}{G}-1}\left(1 - \frac{1}{n}\left(\frac{1}{G}-1\right)\frac{2\beta+1}{2G}\right) + h.o.t.\,, \quad n \gg 6$$

for the topological class distributions $\phi_n = \int \varphi_n(\xi)\, d\xi$.

Proof of Proposition 6.1
Integrating (6.5) w.r.t. ξ and plugging in the ansatz $\phi_n = \lambda^n n^x (1 + \alpha/n)$ we have

$$\begin{aligned} 0 = G\Bigg(&(\beta+1)(n+1)^{x+1}\lambda^{n+1}\left(1+\frac{\alpha}{n+1}\right) - (2\beta+1)\, n^{x+1}\lambda^n\left(1+\frac{\alpha}{n}\right) \\ &+ \beta\,(n-1)^{x+1}\lambda^{n-1}\left(1+\frac{\alpha}{n-1}\right)\Bigg) + n^x\lambda^n\left(1+\frac{\alpha}{n}\right) \end{aligned}$$

considering only $n \geq 6$ such that $\varphi_n(0) = 0$. By formal asymptotics in $1/n$ we find $\lambda = \beta/(\beta+1)$, $x = 1/G - 1$, and $\alpha = -(1/G-1)(2\beta+1)/(2\beta)$ for the parameters in our ansatz. *q.e.d.*

Lemma 6.4 *Assume there exists a smooth solution to* (6.5)*. Let the asymptotic expansion for ϕ_n in Proposition 6.1 be accurate. Then we have*

$$\langle \xi \rangle_n = a\,(n-6) + b + h.o.t.$$

with $a = 1/\,(G+1)$ and $b = a\,((2\beta+1) - 6G)$ for large n.

Proof of Lemma 6.4
Multiplying (6.5) by ξ and integrating yields

$$\begin{aligned} 0 = {} & (n-6)\,\phi_n - G\,(2\beta+1)\,n\,\phi_n\,\langle \xi \rangle_n \\ & + G\,\Big((\beta+1)\,(n+1)\,\phi_{n+1}\,\langle \xi \rangle_{n+1} + \beta\,(n-1)\,\phi_{n-1}\,\langle \xi \rangle_{n-1}\Big) \end{aligned}$$

and dividing by ϕ_n and plugging in the expansion $\phi_n = \lambda^n n^x\,(1+\alpha/n)$ from Proposition 6.1 we have

$$\begin{aligned} 0 = {} & n - 6 - G\,(2\beta+1)\,n\,\langle \xi \rangle_n \\ & + \beta\left(G\,n + 1 + \frac{1}{n}\left(\frac{1}{G} - 1\right)(\beta+1)\right)\langle \xi \rangle_{n+1} \\ & + (\beta+1)\left(G\,n - 1 - \frac{1}{n}\left(\frac{1}{G} - 1\right)\beta\right)\langle \xi \rangle_{n-1} + \text{h.o.t.} \end{aligned}$$

asymptotically. This equation is solved by $\langle \xi \rangle_n = a\,(n-6) + b + o\,(1/n)$ with $a = 1/\,(G+1)$ and $b = a\,(2\beta + 1 - 6G)$. *q.e.d.*

Chapter 7

Conclusion

[The curtain falls.]

We have established a rigorous existence theory for a nonlinear system of transport equations with nonlocal weight that arose from a model for grain growth almost twenty years ago. Now the field is open for further studies concerning self–similar behaviour that we started partially in the last chapter. At this stage it is unclear whether one can succeed with rigorous analytic treatment or if detailed numerical simulations might provide deeper insight.

Besides the use of standard analytic results it was necessary to develop problem specific techniques. Key ideas are the supersolution in Lemma 3.2, the quantile considerations in Lemma 3.4, the “energy” used in Lemma 4.4, and the behaviour of the “quasi–complement” computed in Lemma 5.1.

Using the “quasi–complement” of total mass as a bounding frame growing in time (cf. Theorem 5.1) seems to be — to our knowledge — a new idea in the analysis of infinite–dimensional systems.

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Appendix A

Supplement to the proof of Lemma 4.2
In the sequel we carry out some simple calculations to verify (4.6).

$$
\begin{aligned}
&\sum_{n\geq 2}(n+1)\,f_{n+1}nf_n - n^2f_n^2 + \sum_{n>2}(n-1)\,f_{n-1}nf_n - n^2f_n^2\\
=&\sum_{n\geq 2}(n+1)\,f_{n+1}nf_n - \frac{1}{2}n^2f_n^2 - \frac{1}{2}(n+1)^2\,f_{n+1}^2\\
&+\sum_{n>2}(n-1)\,f_{n-1}nf_n - \frac{1}{2}n^2f_n^2 - \frac{1}{2}(n-1)^2\,f_{n-1}^2\\
=&-\frac{1}{2}\left(\sum_{n\geq 2}((n+1)\,f_{n+1} - nf_n)^2 + \sum_{n\geq 2}((n)\,f_n - (n-1)\,f_{n-1})^2\right)\\
=&-\sum_{n\geq 2}((n+1)\,f_{n+1} - nf_n)^2
\end{aligned}
$$

We are able to produce the desired sign for the symmetric part of the weighted coupling $\sum nf_n\,(Jf)_n$, but while computing the asymmetric part we end up with an additional term $2f_2^2$ due to the shape of $(Jf)_2$.

$$
\begin{aligned}
&\sum_{n>2}(n+1)\,f_{n+1}nf_n - n^2f_n^2 + 3f_3 2f_2\\
=&\sum_{n>2}(n+1)\,f_{n+1}nf_n - \frac{1}{2}n^2f_n^2 - \frac{1}{2}(n+1)^2\,f_{n+1}^2 + 3f_3 2f_2 - \frac{1}{2}3^2f_3^2\\
=&-\frac{1}{2}\sum_{n>2}((n+1)\,f_{n+1} - nf_n)^2 - \frac{1}{2}(3f_3 - 2f_2)^2 + 2f_2^2\\
=&-\frac{1}{2}\sum_{n\geq 2}((n+1)\,f_{n+1} - nf_n)^2 + 2f_2^2
\end{aligned}
$$

The calculations above imply (4.6) finally.

Appendix B

Gradient flow structure of mean curvature flow on triple–junction networks

In this appendix we sketch how to interpret the mean curvature flow on triple–junction networks as a gradient flow. For simplicity, we only treat the isotropic case, i.e. the normal velocity $v = \kappa$ equals (mean) curvature. This is the model used in [9].

Furthermore we indicate how the *Herring condition* arises as a natural boundary condition.

We consider

$$\mathcal{M} := \left\{ \Gamma = \bigcup_{k=1}^{M} \gamma_k \,\middle|\, \text{triple-junction condition} \right\}$$

as a manifold containing all 2d one–periodic networks with lines only meeting in triple junctions. The tangential space attached to each element of the manifold is given by

$$T_\Gamma \mathcal{M} := \left\{ v : \Gamma \to \mathbb{R} \,\middle|\, v \text{ admissible normal velocity} \right\}$$

and the metric tensor reads as

$$g_\Gamma(v, \tilde{v}) := \int_\Gamma v\, \tilde{v}\, dS = \sum_k \int_{\gamma_k} v\, \tilde{v}\, dS$$

for all admissible test velocities $\tilde{v}$. The associated energy is given by

$$E(\Gamma) := \int_\Gamma 1\, dS = \sum_k \int_{\gamma_k} 1\, dS$$

as an L^2 energy. In the sequel we compute the differential of the surface energy on a single grain boundary first

$$
\begin{aligned}
\frac{d}{dt}\int_\gamma \sigma(n,\alpha)\,dS &= \int_0^L \nabla_n\sigma\cdot\partial_t n\;|\partial_s\gamma| + \sigma\frac{\partial_s\gamma}{|\partial_s\gamma|}\cdot\partial_t\partial_s\gamma\,ds \\
&= \int_0^L \left(R^T\nabla_n\sigma + \sigma b\right)\cdot\partial_s\partial_t\gamma\,ds \\
&= -\int_0^L \partial_s\left(R^T\nabla_n\sigma+\sigma b\right)\cdot\partial_t\gamma\,ds + \left(R^T\nabla_n\sigma+\sigma b\right)\cdot\partial_t\gamma\Big|_0^L \\
&= -\int_\gamma \left(\partial^2_{\theta\theta}\sigma+\sigma\right)\kappa n\cdot\tilde{v}\,dS + (\phi n+\psi b)\cdot\tilde{v}\Big|_{\partial\gamma}
\end{aligned}
$$

and by summing up we have

$$
\langle \mathrm{diff}E, \tilde{v}\rangle = -\sum_k\int_{\gamma_k}\left(\partial^2_{\theta\theta}\sigma_k+\sigma_k\right)\kappa n\cdot\tilde{v}\,dS + \frac{1}{2}\sum_k(\phi_k n+\psi_k b)\cdot\tilde{v}\Big|_{\partial\gamma_k}
$$

in a variational formulation. Regarding

$$
-\left\langle \mathrm{diff}E_{\Gamma(t)}, \tilde{v}\right\rangle = g_{\Gamma(t)}(v,\tilde{v})
$$

we observe — in the isotropic case — by plugging in our computations

$$
\sum_k\int_{\gamma_k(t)}\kappa\,\tilde{v}\,dS - \frac{1}{2}\sum_k b\,\tilde{v}\Big|_{\partial\gamma_k(t)} = \sum_k\int_{\gamma_k(t)} v\,\tilde{v}\,dS
$$

that the gradient flow is given by setting $v=\kappa$ if the term $\frac{1}{2}\sum_k b\,\tilde{v}\Big|_{\partial\gamma_k(t)}$ vanishes for all test velocities $\tilde{v}$. Expressing this term by the movement of the end points of the grain boundaries forming a triple junction we require

$$
\frac{1}{2}\sum_k b\,\tilde{v}\Big|_{\partial\gamma_k(t)} = \sum_l\sum_{j=1}^3 b\,\tilde{v}\Big|_{x_l(t)} \overset{!}{=} 0
$$

which leads to the *Herring condition*

$$
\sum_{j=1}^3 b\Big|_{x_l(t)} = 0 \quad \forall l
$$

ensuring local equilibrium of forces at triple junctions.

Appendix C

von Neumann–Mullins law
In the sequel we prove the *von Neumann–Mullins law*

$$\frac{d}{dt}a\left(t\right)=\frac{\pi}{3}\left(n-6\right)$$

relating the change of area with the topological class of a grain.

Let γ_i denote the curves forming the grain boundary, $\vec{v}$ the velocity it moves by, and $\vec{n}$ the outer normal. Furthermore ϑ_i are the tangential angles at the triple junctions and θ_i the inner angles between two curves at a triple junction.

$$\begin{aligned}\frac{d}{dt}a\left(t\right)&=\sum_{i=1}^{n}\int_{\gamma_i}\vec{v}\cdot\vec{n}\,dS=\sum_{i=1}^{n}\int_{\gamma_i}\kappa\,dS\\&=-\sum_{i=1}^{n}\int_{0}^{l_i}\partial_s\vartheta_i\,ds=\sum_{i=1}^{n}\vartheta_i\left(0\right)-\vartheta_i\left(l_i\right)\\&=\sum_{i=1}^{n}\vartheta_{i+1}\left(0\right)-\vartheta_i\left(l_i\right)-2\pi=\sum_{i=1}^{n}\left(\pi-\theta_i\right)-2\pi\\&=\sum_{i=1}^{n}\left(\pi-\frac{2}{3}\pi\right)-2\pi=\frac{\pi}{3}\left(n-6\right)\end{aligned}$$

We used that grain boundaries move by mean curvature, i.e. $\vec{v}\cdot\vec{n}=\kappa$, and the convention $\vartheta_{i+n}=\vartheta_i+2\pi$ reflecting that we move along a closed curve. Plugging in the prescribed angle condition $\theta_i=2\pi/3$ at triple junctions completes the proof.

Danksagung

Die vorliegende Arbeit entstand während der letzten drei Jahre am

Institut für Mathematik der Humboldt–Universität zu Berlin

im Rahmen des Projekts

Homogenization of many–particle systems

am Lehrstuhl für Angewandte Analysis. Das Projekt innerhalb des Schwerpunktprogramms 1095

Analysis, Modeling, and Simulation of Multiscale Problems

wurde von der Deutschen Forschungsgemeinschaft finanziell gefördert.

Mein Dank gilt an erster Stelle Prof. Dr. Barbara Niethammer für die ständige Begleitung dieses Projekts. Sie bot mir immer die Gelegenheit, auftretende Fragen zeitnah zu erörtern. Wesentliche methodische Ansätze in der Analysis partieller Differentialgleichungen habe ich von ihr gelernt.

Weiterhin möchte ich Prof. Dr. Juan J. L. Velázquez für die intensiven und erbauenden Diskussionen in Leipzig und Berlin danken. Sein stetiges Interesse, grundlegende Vorgänge in kinetischen Modellen zu verstehen, war mir immer Vorbild.

Dr. Michael Herrmann war in doppelter Hinsicht ein wertvoller Gesprächspartner. Ich konnte sowohl wichtige Ideen als auch Schwierigkeiten technischer Art mit ihm diskutieren; dafür bin ich sehr dankbar.

Priv.-Doz. Dr. Lutz Recke gab mir wertvolle Hinweise und Anregungen zur Darstellung der Ergebnisse.

Schließlich konnte ich auch mit weiteren Kollegen in Berlin und Bonn diskutieren und so meine Argumente auf Verständlichkeit prüfen. Stellvertretend seien hier Sven–Joachim Kimmerle und Rubén Cantero–Álvarez genannt.

Der Gruppe um Prof. Dr. Felix Otto in Bonn möchte ich für ihre herzliche Gastfreundschaft danken.

Lebenslauf

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Selbständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation

A kinetic model for grain growth

selbständig und ohne unerlaubte Hilfe angefertigt habe.

Reiner Henseler